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Monetary Policy with Model Uncertainty: Distribution Forecast Targeting

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Abstract

We examine optimal and other monetary policies in a linear-quadratic setup with a relatively general form of model uncertainty. The form of model uncertainty our framework encompasses includes: simple i.i.d. model deviations; serially correlated model deviations; estimable regime-switching models; more complex structural uncertainty about very different models, for instance, backward- and forward-looking models; time-varying central-bank judgment about the state of model uncertainty; and so forth. We provide an algorithm for finding the optimal policy as well as solutions for arbitrary policy functions. This allows us to compute and plot consistent distribution forecasts—fan charts—of target variables and instruments. Our methods hence extend certainty equivalence and "mean forecast targeting" to more general certainty non-equivalence and "distribution forecast targeting."

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Contents

1 Introduction

In recent years there has been a renewed interest in the study of optimal monetary policy under uncertainty. Classical analysis of optimal policy consider only additive sources of uncertainty, where in a linear-quadratic framework the well-known certainty-equivalence result applies and implies that optimal policy is the same as if there were no uncertainty. Recognizing the uncertain environment that policymakers face, recent research has considered broader forms of uncertainty for which certainty equivalence no longer applies. While this may have important implications, in practice the design of policy becomes much more difficult outside the classical linear-quadratic framework.

In this paper we develop a relatively general form of model uncertainty that remains quite tractable. Our approach allows us to move beyond the classical linear-quadratic world with additive shocks, yet remains close enough to the linear-quadratic framework that the analysis is transparent. We examine optimal and other monetary policies in an extended linear-quadratic setup, extended in a way to capture model uncertainty. The form of model uncertainty our framework encompasses includes: simple i.i.d. model deviations; serially correlated model deviations; estimable regime-switching models; more complex structural uncertainty about very different models, for instance, backward- and forward-looking models; time-varying central-bank judgment about the state of model uncertainty; and so forth. We provide an algorithm for finding the optimal policy as well as solutions for arbitrary policy functions. This allows us to compute and plot consistent distribution forecasts—fan charts—of target variables and instruments. Our methods hence extend certainty equivalence and "mean forecast targeting" (Svensson [16]) to more general certainty non-equivalence and "distribution forecast targeting."

In section 2, we lay out the model, a so-called Markov jump-linear-quadratic (MJLQ) model, where model uncertainty takes the form of different "modes" that follow a Markov process. We extend existing MJLQ models to incorporate forward-looking variables, the existence of which makes the model nonrecursive. We show that the recursive saddlepoint method of Marcet and Marimon [11] can nevertheless be applied to express the model in a convenient recursive way, derive an algorithm for determining the optimal policy and value functions, and discuss how different kinds of model uncertainty are incorporated by our framework. In section 4, we present examples based on two empirical models of the US economy: regime-switching versions of the backward-looking model of Rudebusch and Svensson [13] and the forward-looking New Keynesian model of Lindé [9]. In section 5, we show how probability distributions of forecasts—fan charts—of relevant variables can be constructed for arbitrary time-varying instrument-rate paths or functions. In section 6, we show how the same probability distributions can be constructed for arbitrary time-invariant instrument rules and optimal restricted instrument rules. In section 7, we show how the optimal policy and value functions can be expressed as a function of the probability distribution of the modes, when these modes are not observed. In section 8 [to be added], we present some conclusions. Appendices A-E contain some technical details.

2 The model and its interpretation

We set up a relatively flexible model of an economy with a central bank, which allows for serially correlated additive and multiplicative uncertainty as well as different relevant representations of the central bank's information and judgment about the economy.[1]

2.1 The baseline model

Consider the following model of an economy with a central bank,

$$X_{t+1} = A_{11,t+1}X_t + A_{12,t+1}x_t + B_{1,t+1}i_t + C_{t+1}\varepsilon_{t+1} \tag{2.1}$$

$$\mathrm{E}_t H_{t+1}x_{t+1} = A_{21,t}X_t + A_{22,t}x_t + B_{2,t}i_t, \tag{2.2}$$

where X_t is an n_X-vector of predetermined variables in period t, x_t is an n_x-vector of forward-looking variables in period t, i_t is an n_i-vector of central-bank instruments (control variables) in period t, and ε_t is a zero-mean i.i.d. shock realized in period t with covariance matrix $\sigma^2 I$. The matrix $A_{22,t}$ is nonsingular, so equation (2.2) determines the forward-looking variables in period t. There is no restriction in including the shock ε_t only in the equations for the predetermined variables, since, if necessary, the set of predetermined variables can always be expanded to include the shocks and this way enter into the equations for the forward-looking variables. The expression $\mathrm{E}_t w_{t+1}$ denotes the conditional expectation in period t of a random variable w_{t+1} realized in period $t+1$. The information assumption for the conditional expectations operator E_t is specified below.

The central bank has an intertemporal loss function in period t,

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau}, \tag{2.3}$$

[1] As shown in appendix A, our framework can also incorporate additive central-bank judgment as in Svensson [16].

where the period loss, L_t, satisfies

$$L_t \equiv \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}' W_t \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} \tag{2.4}$$

and the matrix W_t is symmetric and positive semidefinite.

The matrices $A_{11,t}$, $A_{12,t}$, $B_{1,t}$, C_t, H_t, $A_{21,t}$, $A_{22,t}$, $B_{2,t}$, and W_t (assumed to be of appropriate dimension) are random and can each take n different values in period t, corresponding to the n *modes* $j_t = 1, 2, ..., n$ in period t. We denote these values $A_{11,t} = A_{11j_t}$, $A_{12,t} = A_{12j_t}$, and so forth, for $j_t = 1, 2, ..., n$. The modes j_t follow a Markov process with constant transition probabilities

$$P_{jk} \equiv \Pr\{j_{t+1} = k \mid j_t = j\} \qquad (j, k = 1, ..., n). \tag{2.5}$$

Importantly, the shocks ε_t and the modes j_t are assumed to be independently distributed (although we allow the impact on the economy of the shocks to depend on the modes j_t through the matrix C_{j_t}). Furthermore, P denotes the $n \times n$ transition matrix $[P_{jk}]$, $p_{jt} \equiv \Pr\{j_t = j\}$ $(j = 1, ..., n)$, and $p_t \equiv (p_{1t}, ..., p_{nt})'$ denotes the probability distribution of the modes in period t, so

$$p_{t+1} = P' p_t.$$

Finally, $\bar{p}$ denotes the stationary distribution of the modes, so

$$\bar{p} = P' \bar{p}.$$

In the beginning of period t, before the central bank chooses the instruments, i_t, the central bank's information set includes the realizations of X_t, j_t, ε_t, X_{t-1}, j_{t-1}, ε_{t-1}, x_{t-1}, i_{t-1}, ... The central bank also knows the probability distribution of the innovation ε_t, the transition matrix P, and the n different values each matrices can take. Hence, the conditional expectations operator, E_t, refers to expectations conditional on that information. In section 7 we consider the alternative and more realistic situation when the mode j_t is not observed in period t and policy in period t is based on the probability distribution p_t of the modes.

We consider the optimization problem of minimizing (2.3) subject to (2.4), (2.1), (2.2), and X_t given. In particular, we consider the optimization under commitment in a timeless perspective (see Woodford [19] and Svensson and Woodford [18]).

Optimization problems of this type have been studied in the control-theory literature for the special case when there are no forward-looking variables. Such models are known as Markov jump-linear-quadratic (MJLQ) systems, as the model is conditionally linear but operates in multiple modes which are governed by a Markov jump process. Such MJLQ systems have been widely studied in the control-theory literature in the last few years (see Costa and Fragoso [3], Costa, Fragoso, and Marques [4], do Val, Geromel, and Costa [6], and the references therein).

2.2 Reformulation according to the recursive saddlepoint method

In order to apply the methods developed in control theory, we require that the system be recursive. However, the presence of the forward-looking variables in (2.2) makes the problem nonrecursive. Fortunately, the recursive saddlepoint method of Marcet and Marimon [11] can be applied to reformulate the non-recursive problems with forward-looking variables as recursive saddlepoint problem (see Marcet and Marimon [11] and Svensson [17] for details).

The problem of minimizing the intertemporal loss function in each period under commitment in a timeless perspective can be reformulated as the modified saddlepoint problem,

$$\max_{\{\gamma_{t+\tau}\}_{\tau\geq 0}} \min_{\{x_{t+\tau}, i_{t+\tau}\}_{\tau\geq 0}} \mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau \tilde{L}_{t+\tau}, \tag{2.6}$$

with the modified period loss function,

$$\tilde{L}_{t+\tau} \equiv \begin{bmatrix} \tilde{X}_{t+\tau} \\ \tilde{\imath}_{t+\tau} \end{bmatrix}' \tilde{W}_{j_{t+\tau}} \begin{bmatrix} \tilde{X}_{t+\tau} \\ \tilde{\imath}_{t+\tau} \end{bmatrix}, \tag{2.7}$$

subject to the modified model

$$\tilde{X}_{t+\tau+1} = \tilde{A}_{j_{t+\tau+1}} \tilde{X}_{t+\tau} + \tilde{B}_{j_{t+\tau+1}} \tilde{\imath}_{t+\tau} + \tilde{C}_{j_{t+\tau+1}} \varepsilon_{t+\tau+1} \tag{2.8}$$

for $\tau \geq 0$, where $\tilde{X}_t$ and j_t are given. Here, the new $n_{\tilde{X}}$-vector of predetermined variables $\tilde{X}_t$ ($n_{\tilde{X}} \equiv n_X + +n_x$) and the new $n_{\tilde{\imath}}$-vector of instruments $\tilde{\imath}_t$ ($n_{\tilde{\imath}} \equiv n_x + n_i + n_x$) are defined as

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad \tilde{\imath}_t = \begin{bmatrix} x_t \\ i_t \\ \gamma_t \end{bmatrix}. \tag{2.9}$$

The elements of the n_x-vector Ξ_{t-1} are the Lagrange multipliers for the equations (2.2) for the forward-looking variables in period $t-1$ from the optimization problem in that period. Hence, Ξ_{t-1} captures the history dependence of the optimal policy under commitment in a timeless perspective (see Woodford [19] and Svensson and Woodford [18]). The elements of the n_x-vector γ_t are the Lagrange multipliers for equations (2.2) in period t, considered as control variables in period t. Hence, we have

$$\Xi_t = \gamma_t$$

as an additional dynamic equation, which is incorporated in (2.8).

The matrix $\tilde{W}_{j_t}$ in (2.7) is constructed so the modified period loss $\tilde{L}_t$ satisfies

$$\tilde{L}_t \equiv L_t + \gamma_t'(-A_{21j_t}X_t - A_{22j_t}x_t - B_{2j_t}i_t) + \frac{1}{\delta}\Xi_{t-1}' H_{j_t}x_t. \tag{2.10}$$

The matrices $\tilde{A}_{j_{t+1}}$, $\tilde{B}_{j_{t+1}}$, and $\tilde{C}_{j_{t+1}}$ satisfy

$$\tilde{A}_{j_{t+1}} \equiv \begin{bmatrix} A_{11j_{t+1}} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B}_{j_{t+1}} \equiv \begin{bmatrix} A_{12j_{t+1}} & B_{1j_{t+1}} & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix}, \qquad \tilde{C}_{j_t} \equiv \begin{bmatrix} C_{j_{t+1}} \\ 0 \end{bmatrix}. \tag{2.11}$$

2.3 Optimal policy and dynamics

The solution of the modified saddlepoint problem will result in a linear optimal policy function

$$\tilde{\imath}_t = F_{j_t}\tilde{X}_t \qquad (j_t = 1, ..., n) \tag{2.12}$$

and a modified quadratic value function

$$\tilde{X}_t \tilde{V}_{j_t} \tilde{X}_t + \tilde{w}_{j_t} = \max_{\{\gamma_{t+\tau}\}_{\tau\geq 0}} \min_{\{x_{t+\tau}, i_{t+\tau}\}_{\tau \geq 0}} \mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau \tilde{L}_{t+\tau}, \qquad (j_t = 1, ..., n) \tag{2.13}$$

(see appendix B for details and a convenient algorithm for computing V_j and F_j for $j = 1, ..., n$). The optimal policy function for the modified problem is also the solution to the original problem.

Consider the composite state $(\tilde{X}_t, j_t)$ in period t, where $\tilde{\imath}_t = F_{j_t}\tilde{X}_t$. The transition from this composite state to the composite state $(\tilde{X}_{t+1}, j_{t+1})$ in period $t+1$ with $\tilde{\imath}_{t+1} = F_{j_{t+1}}\tilde{X}_{t+1}$ will satisfy

$$\tilde{X}_{t+1} = M_{j_t j_{t+1}}\tilde{X}_t + \tilde{C}_{j_{t+1}}\varepsilon_{t+1},$$

where

$$M_{j_t j_{t+1}} \equiv \tilde{A}_{j_{t+1}} + \tilde{B}_{j_{t+1}}F_{j_t},$$

and will, for given realization of ε_{t+1}, occur with probability $P_{j_t j_{t+1}}$. This determines the optimal distribution of future $\tilde{X}_{t+\tau}$, $j_{t+\tau}$, and $\tilde{\imath}_{t+\tau}$ $(\tau \geq 1)$ conditional on $(\tilde{X}_t, j_t)$.

Such conditional distributions can be illustrated by plots of future means, medians, and percentiles (fan charts). Plots of future means, medians, and percentiles can also be constructed for individual chains of the modes, for instance, the median or mean chain corresponding to no model uncertainty.

Note that the above value function is the value function corresponding to the modified period loss function and the modified saddlepoint problem. The value function for the original problem of minimizing (2.3) subject to (2.1), (2.2), and (2.4) under commitment in a timeless perspective with $\tilde{X}_t$ given is

$$\tilde{X}_t' V_{j_t} \tilde{X}_t + w_{j_t}.$$

The matrices V_j and the scalars w_j for $j = 1, ..., n$, are determined in the following way:

Let F_{j_t} be decomposed conformably with x_t, i_t, and γ_t,

$$F_{j_t} \equiv \begin{bmatrix} F_{xj_t} \\ F_{ij_t} \\ F_{\gamma j_t} \end{bmatrix},$$

and note that we have

$$\begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{F_{xj_t}} \\ \multicolumn{2}{c}{F_{ij_t}} \end{bmatrix} \tilde{X}_t.$$

It follows that we can write the period loss function as

$$L_t = \tilde{X}_t' \bar{W}_{j_t} \tilde{X}_t,$$

where

$$\bar{W}_{j_t} \equiv \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{F_{xj_t}} \\ \multicolumn{2}{c}{F_{ij_t}} \end{bmatrix}' W_{j_t} \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{F_{xj_t}} \\ \multicolumn{2}{c}{F_{ij_t}} \end{bmatrix}. \tag{2.14}$$

The matrix V_j will then satisfy the Lyapunov function

$$V_j = \bar{W}_j + \delta \sum_k P_{jk} M_{jk}' V_k M_{jk}, \tag{2.15}$$

and the constant w_j will satisfy the equation[2]

$$w_j = \delta \sum_k P_{jk} [\mathrm{tr}(V_k \sigma^2 \tilde{C}_k \tilde{C}_k') + w_k]. \tag{2.16}$$

3 Interpretation of model uncertainty in our framework

The assumptions that the random matrices of coefficients take a finite number of values corresponding to a finite number of modes and that these modes follow a Markov process independent of the additive innovations allow us to use the convenient and flexible framework of MJLQ systems—once we apply the recursive saddlepoint method of Marcet and Marimon to reformulate the non-recursive model with forward-looking variables as a recursive model. By specifying different configurations of modes and transition probabilities, we can approximate many different kinds of model uncertainty.

[2] Note that $\sigma^2 \tilde{C}_k \tilde{C}_k'$ is the covariance matrix of the shocks $\tilde{C}_k \varepsilon_{t+1}$ to $\tilde{X}_{t+1}$ when $j_{t+1} = k$ $(k = 1, ..., n)$.

- Both i.i.d. and serially correlated random coefficients of the model can be handled.

- The modes can correspond to different structural models. The models can differ by having different relevant variables, different number of leads or lags, or the same variable being predetermined in one model and forward-looking in another. Thus, one mode can represent a model with forward-looking variables such as the New Keynesian model of Lindé [9], another a backward-looking model such as that of Rudebusch and Svensson [13] (see appendix C for details).

- The modes can correspond to situations when variables such as inflation and output have more or less inherent persistence (are more or less autocorrelated), when the exogenous shocks have more or less persistence (add a new predetermined variables equal to the serially correlated shock, and let this new predetermined variable be an AR(1) process with a high or low coefficient), or when the uncertainty about the coefficients or models are higher or lower.

- The modes can be structured such that they correspond to different central-bank judgments about model coefficients and model uncertainty. Let $j_t = 1, ..., n$ correspond to n different *model* modes (different coefficients, different variance or persistence of coefficient disturbances, or different variance of the ε_t shocks (via different matrices C_j)). Let $k_t = 1, ..., m$ correspond to m different central-bank *judgment* modes. Let each judgment mode correspond to some central-bank information about the model modes. This can generally be modeled as a situation where the transition matrix for the model modes, $\tilde{P}$, depends on the judgment mode. Let the transition matrix for model modes be $\tilde{P}(k_t)$, for $k_t = 1, ..., m$, and hence depend on k_t. Let P^0 denote the transition matrix for the judgment modes (assumed independent of the model modes). We can then consider a composite model-judgment mode (j_t, k_t) in period t, with the transition probability from model-judgment mode (h, k) in period t to mode (j, l) in period $t + 1$ given by $\tilde{P}(k)_{hj}P^0_{kl}$. For instance, the judgment modes may correspond to different persistence of the model modes.

- The mode j_t may be observed in period t, in which case optimal policy and the value function is conditional on the mode j_t. Alternatively, and more realistically, we may assume that the mode is not perfectly observed. Then we can represent the central bank's information in period t about the mode as the distribution p_t of the modes. Then optimal policy and the value function in period t will depend on the distribution p_t.

- As noted in appendix A, we can combine multiplicative uncertainty about the modes with the additive uncertainty about future deviations. This way we can simultaneously handle central-bank judgment about future additive deviations as in Svensson [16] and central-bank judgment about model modes as in this paper. For instance, we can handle situations when there is more or less uncertainty about shocks farther into the future relative those in the near future.

Generally, aside from dimensional and computational limitations, it is difficult to conceive of a situation for a policymaker that cannot be approximated in this framework.

4 Examples

In this section, we present examples based on two empirical models of the US economy: regime-switching versions of the backward-looking model of Rudebusch and Svensson [13] and the forward-looking New Keynesian model of Lindé [9].

4.1 An estimated backward-looking model

In this section we consider the effects of serially correlated parameter variation in the quarterly model of the US economy of Rudebusch and Svensson [13], henceforth RS. Using the same data set as in their paper, we estimate a three-mode MJLQ (or Markov-switching) model using the Bayesian Gibbs sampling methods described in Kim and Nelson [8], and we compare the implications to the constant-coefficient version of RS.[3]

The key variables in the model are quarterly annualized inflation π_t, the output gap y_t, and the interest rate (the federal funds rate) i_t. The model has the following form:

$$\begin{aligned} \pi_{t+1} &= \sum_{\tau=0}^{2} \alpha_{\tau j}\pi_{t-\tau} + \left(1 - \sum_{\tau=0}^{2} \alpha_{\tau j}\right)\pi_{t-3} + \alpha_{4j} y_t + \varepsilon_{\pi,t+1}, \\ y_{t+1} &= \beta_{1j} y_t + \beta_{2j} y_{t-1} + \beta_{3j}\left(\bar{\imath}_t - \bar{\pi}_t\right) + \varepsilon_{y,t+1}, \end{aligned} \tag{4.1}$$

where $j \in \{1,2,3\}$ indexes the mode, $\bar{\imath}_t \equiv \sum_{\tau=0}^{3} i_{t-\tau}/4$ and $\bar{\pi}_t \equiv \sum_{\tau=0}^{3} \pi_{t-\tau}/4$ are 4-quarter averages, and the shocks $\varepsilon_{\pi t}$ and ε_{yt} are distributed $N(0,\sigma_{\pi j}^2)$ and $N(0,\sigma_{yj}^2)$, respectively.

Table 4.1 reports our estimates of the posterior mean, with the estimates from the constant-parameter version of the model for comparison. Details of the estimation method and prior setting

[3] The estimation is preliminary and will be refined in future revisions.

Parameter	Constant	Mode 1	Mode 2	Mode 3
α_0	0.6922	0.7076	0.6509	0.7090
α_1	−0.1033	−0.0864	−0.0892	−0.0934
α_2	0.2786	0.2188	0.2990	0.2312
α_3	0.1021	0.1600	0.1393	0.1532
β_1	1.1591	1.1040	1.1731	1.1373
β_2	−0.2521	−0.1980	−0.2845	−0.2362
β_3	−0.0990	−0.1598	−0.0699	−0.0677
σ_π	1.0090	1.6279	3.3033	1.1764
σ_y	0.8190	1.0490	2.1371	0.7971

Table 4.1: Estimates of the constant-parameter and three-mode Rudebusch-Svensson model.

are in an appendix (to be written). Here we see that many of the coefficient estimates are quite stable across modes. The biggest differences come in the slope of the IS curve β_3, which is largest in magnitude in mode 1, and in the sizes of the shocks, where we see that mode 2 is roughly twice as volatile as the other modes. The estimated transition matrix P and its implied stationary distribution $\bar{p}$ are

$$P = \begin{bmatrix} 0.8911 & 0.0072 & 0.1017 \\ 0.0029 & 0.9425 & 0.0547 \\ 0.1098 & 0.0074 & 0.8827 \end{bmatrix}, \qquad \bar{p} = \begin{bmatrix} 0.4473 & 0.1123 & 0.4404 \end{bmatrix}.$$

Thus mode 2 is the most volatile and has the smallest mass in the invariant distribution.

The estimated probabilities of being in the different modes are shown in figure 4.1. The plots show both the filtered estimates, in which the distribution in period t is only estimated by using data up to t, as well as the smoothed estimates, in which the distribution in period t is estimated with data for the whole sample. Clearly, there are more fluctuations in the filtered estimates than in the smoothed ones, since by looking backward we can better assess the chance of being in a particular regime. We see that for most of the sample, the economy was assessed to be in either modes 1 or 3, where the volatile mode 2 only shows significant probability around 1975.

Model/Mode	π_t	π_{t-1}	π_{t-2}	π_{t-3}	y_t	y_{t-1}	i_{t-1}	i_{t-2}	i_{t-3}
Constant	0.9921	0.3465	0.4273	0.1381	1.7974	−0.4639	0.3713	−0.0899	−0.0456
Mode 1	1.2350	0.4422	0.5240	0.2361	1.6501	−0.3480	0.2813	−0.1205	−0.0626
Mode 2	1.1634	0.4434	0.5378	0.1854	1.7698	−0.5240	0.4680	−0.0634	−0.0314
Mode 3	1.1965	0.3820	0.4831	0.2098	1.6910	−0.3937	0.4095	−0.0749	−0.0360

Table 4.2: Optimal policy functions for the constant parameter and three-mode Rudebusch-Svensson model.

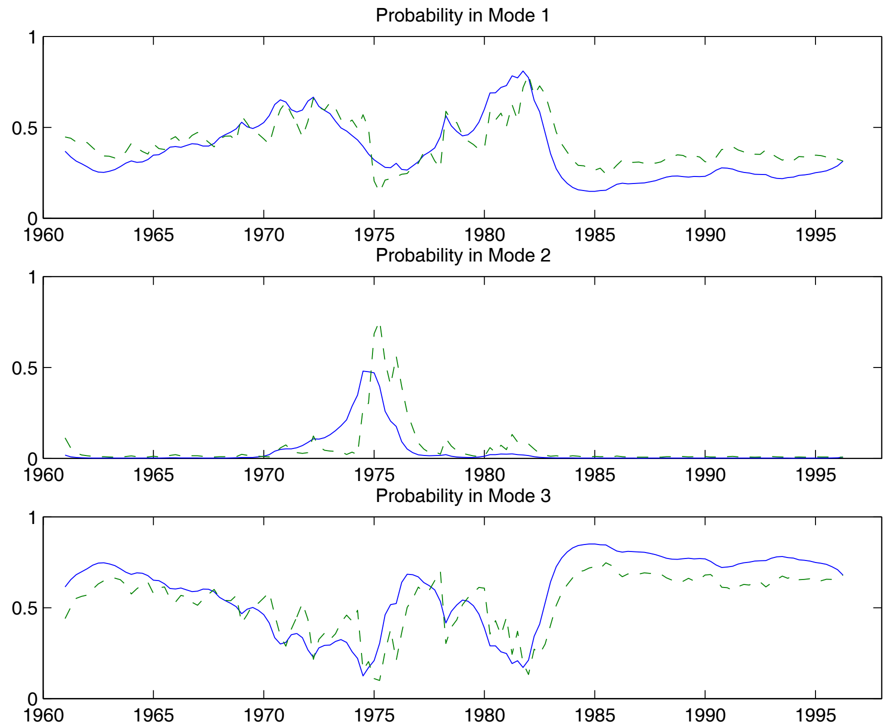

Figure 4.1: Estimated probabilities of being the different modes. Smoothed (full-sample) inference is shown with solid lines, while filtered (one-sided) inference is shown with dashed lines.

We let the period loss function be

$$L_t = \frac{1}{2}[\pi_t^2 + \lambda y_t^2 + \nu(i_t - i_{t-1})^2] \tag{4.2}$$

with the parameters $\lambda = 1$, $\nu = 0.2$, and $\delta = 1$ (δ is the discount factor in the intertemporal loss function, (2.3)). We then solve for the optimal policy function,

$$i_t = F_j X_t,$$

where $X_t = (\pi_t, \pi_{t-1}, \pi_{t-2}, \pi_{t-3}, y_t, y_{t-1}, i_{t-1}, i_{t-2}, i_{t-3})'$, using the methods described above. The optimal policy functions are given in table 4.2. In figure 4.2, we plot the average impulse responses of inflation, the output gap, and the interest rate to the two shocks in the model. In particular, for 10,000 simulation runs we first draw an initial mode of the Markov chain from its stationary distribution, then simulate the chain for 50 periods forward, tracing out the impulse responses. The figure plots the median response at each date, along with 5% and 95% quantiles of the empirical distribution. Also shown for comparison are the responses under the optimal policy for the constant-coefficient estimates given above.

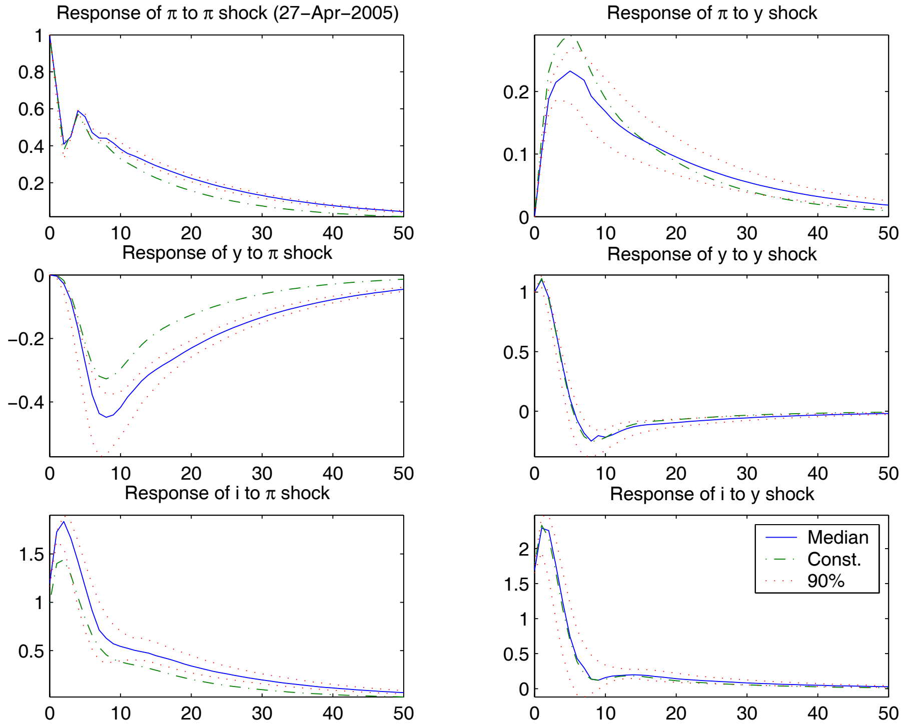

Figure 4.2: Unconditional impulse responses to shocks under the optimal policy for the Rudebusch-Svensson model. Shown are the median response (solid lines) and the 90% probability bands (dotted), along with the optimal responses with constant coefficients (dashed).

Both the table and the figure illustrate that the Markov uncertainty leads to a change in the nature of policy. In all three modes the optimal policy is more aggressive in response to inflation in the MJLQ setting versus the constant coefficient optimal rule. However in response to a shock to the output gap, the optimal policy is nearly the same as in the constant coefficient case. This suggests that uncertainty about the Phillips curve coefficients may be more important in this model.

4.2 An estimated forward-looking model[4]

We now consider the effects of uncertainty in a model with both forward- and backward-looking elements. We use the same data set as above, and again estimate a three-mode MJLQ model. The

[4] This section reports the results of a preliminary maximum-likelihood estimation. A more thorough Bayesian estimation is in progress.

Parameter	Constant	Mode 1	Mode 2	Mode 3
ω_f	0.5187	0.4920	0.5454	0.5214
γ	0.0020	0.0001	0.0017	0.0032
β_f	0.4467	0.3241	0.4523	0.4838
β_r	0.0062	0.0154	0.0038	0.0034
β_y	1.2215	1.4357	1.3326	1.0337
ρ_1	0.9861	0.5901	0.9502	1.1285
ρ_2	−0.0849	−0.1496	0.0369	−0.2071
γ_π	1.4824	1.5797	1.1056	1.2002
γ_y	0.9463	0.2217	1.6165	0.8691
σ_π	0.5761	0.3408	1.0050	0.4575
σ_y	0.3871	0.5630	0.4624	0.2673
σ_i	1.0023	1.1480	1.2006	0.3026

Table 4.3: Estimates of the constant parameter and three-mode Lindé model.

structural model is a simplification of the model of the US economy in Lindé [9] and is given by

$$\begin{aligned}
\pi_t &= \omega_{fj} E_t \pi_{t+1} + (1-\omega_{fj})\pi_{t-1} + \gamma_j y_t + \varepsilon_{\pi t}, \qquad (4.3)\\
y_t &= \beta_{fj} E_t y_{t+1} + (1-\beta_{fj}) \left[\beta_{yj} y_{t-1} + (1-\beta_{yj}) y_{t-2}\right] - \beta_{rj} \left(i_t - E_t \pi_{t+1}\right) + \varepsilon_{yt},\\
i_t &= \left(1-\rho_{1j}-\rho_{2j}\right)\left(\gamma_{\pi j}\pi_t + \gamma_{yj} y_t\right) + \rho_{1j} i_{t-1} + \rho_{2j} i_{t-2} + \varepsilon_{it}.
\end{aligned}$$

where again $j \in \{1,2,3\}$ indexes the mode, and the shocks $\varepsilon_{\pi t}$, ε_{yt}, and ε_{it} are normally distributed with zero means and variances $\sigma^2_{\pi j}$, σ^2_{yj}, and σ^2_{ij}, respectively.

Table 4.3 reports our estimates, with the estimates from the constant-parameter version of the model for comparison. Our constant-parameter estimates are similar to those in Lindé [9], with the main difference that we find much smaller estimates of γ and β_r. We also see that many of the key structural parameters change relatively little across modes, while the policy-function coefficients and shock standard deviations display much more variability. For example, mode 2 has relatively large inflation shocks and a relatively large policy response to the output gap. Mode 3 on the other hand has a super-inertial policy response to the lagged interest rate, coupled with relatively small shocks to all variables. The estimated transition matrix P and its implied stationary distribution $\bar{p}$ are given by:

$$P = \begin{bmatrix} 0.9885 & 0.0049 & 0.0065 \\ 0.0333 & 0.8650 & 0.1016 \\ 0.0039 & 0.0108 & 0.9854 \end{bmatrix} \qquad \bar{p} = \begin{bmatrix} 0.3658 & 0.0593 & 0.5749 \end{bmatrix}$$

Thus modes 1 and 3 are quite persistent, and have the largest mass in the invariant distribution.

The estimated probabilities of being in the different modes are shown in figure 4.3. Again the plots show both the smoothed and filtered estimates. We see that mode 1 was unlikely to be

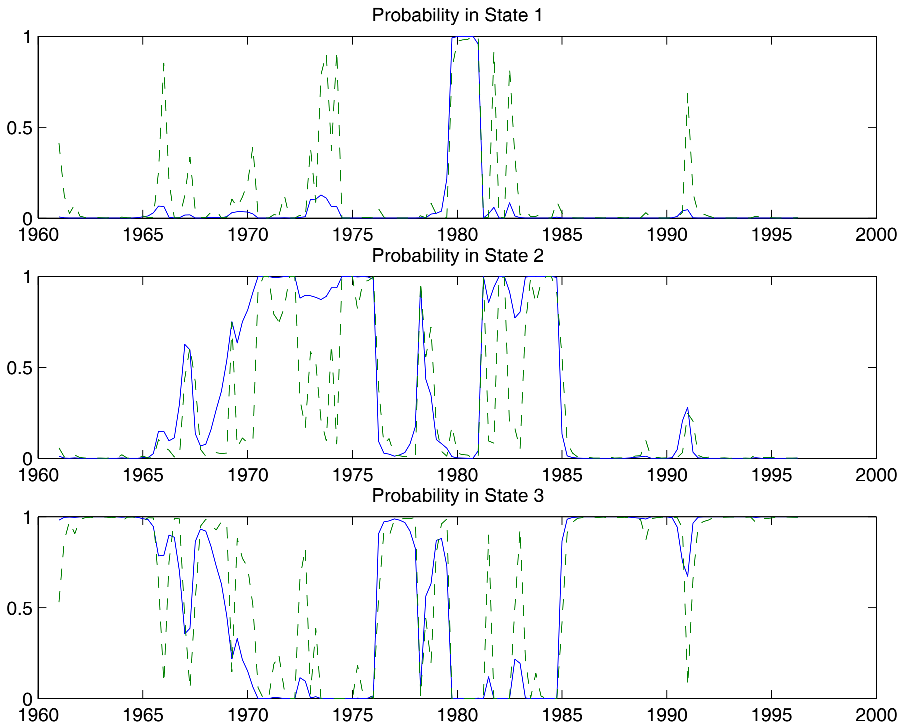

Figure 4.3: Estimated probabilities of being the different modes. Smoothed (full-sample) inference is shown with solid lines, while filtered (one-sided) inference is shown with dashed lines.

Model/Mode	π_{t-1}	y_{t-1}	y_{t-2}	$\varepsilon_{\pi t}$	ε_{yt}	$\Xi_{\pi,t-1}$	$\Xi_{y,t-1}$
Constant	-0.9146	1.3792	0.0388	-0.4490	2.0407	0.0005	0.0411
Mode 1	-0.8641	1.9232	-0.4429	-0.0696	1.9819	0.0000	0.0216
Mode 2	-0.9533	1.5144	-0.1579	-0.3984	2.0750	0.0003	0.0389
Mode 3	-0.8212	0.9970	0.5740	-0.9068	1.8688	0.0011	0.0343

Table 4.4: Optimal policy functions of the constant parameter and three-mode Lindé model.

experienced throughout much of the sample, only showing significant probability around 1980. The 1970s and early 1980s were a period when mode 2 was predominant, while mode 3 was predominant before 1970 and after 1985.

We again solve for the optimal policy function,

$$i_t = F_{ij}\tilde{X}_t,$$

where $\tilde{X}_t \equiv (\pi_{t-1}, y_{t-1}, y_{t-2}, \varepsilon_{\pi t}, \varepsilon_{yt}, \Xi_{\pi,t-1}, \Xi_{y,t-1})'$, using the methods described above. The optimal policy functions are given in table 4.4, and in figure 4.4 we plot the average impulse responses of inflation, the output gap, and the interest rate to the structural two shocks in the model. Again we consider 10,000 simulations of 50 periods, and plot the median responses along with the 90%

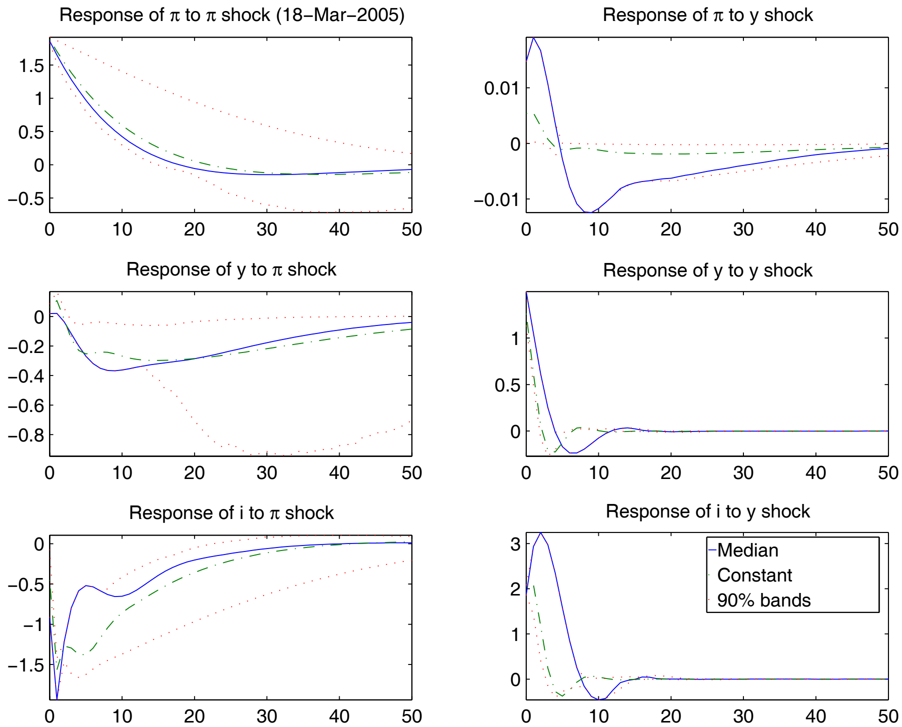

Figure 4.4: Unconditional impulse responses to shocks under the optimal policy for the Lindé model. Shown are the median response (solid lines) and the 90% probability bands (dotted), along with the optimal responses with constant coefficients (dashed).

bands and the corresponding optimal responses for the constant-coefficient estimates.

Again the Markov uncertainty leads to a dramatic change in the nature of policy. This is most noticeable in the response to a shock to the output gap, where the optimal policy response is more aggressive and longer-lived than in the constant-coefficient case. Interestingly, there is relatively little variation in the distribution of responses to this shock, but significantly variation in the responses to an inflation shock. Although the median responses track the constant-coefficient version here, there is significant dispersion.

5 Arbitrary time-varying instrument path rules

In this section we derive the dynamics of the system, including the distribution of forecasts of relevant future variables, for arbitrary time-varying instrument rules, including time-varying instrument paths such as a constant interest rate for arbitrary (but finitely many) periods. We also specify the optimization problem for instrument rules in a given class of instrument rules.

Consider implementing an arbitrary time-varying instrument rule during period $t = 0, 1, ..., T-1$ and implementing the optimal policy function from period T on. Let the arbitrary instrument rule be linear but otherwise of the rather general form

$$i_t = F_{\tilde{X}tj_t}\tilde{X}_t + F_{xtj_t}x_t \qquad (0 \leq t \leq T-1), \tag{5.1}$$

where $\tilde{X}_t$ denotes the $n_{\tilde{X}}$-vector $(X_t', \Xi_{t-1}')'$, $F_{\tilde{X}tj_t}$ and F_{xtj_t} are $(n_i \times n_{\tilde{X}})$ and $(n_i \times n_x)$ matrices, respectively, which depend on both the period t and the mode j_t. For added generality, we also allow a response to the forward-looking variables, x_t.

If $F_{xtj_t} \equiv 0$, this is an *explicit* instrument rule; that is, the instrument responds to predetermined variables only (policy functions and explicit instrument rules are the same. If $F_{xtj_t} \not\equiv 0$ ($F_{xtj_t} \neq 0$ for some mode j_t with positive probability), it is an *implicit* instrument rule; that is, the instrument depends also on forward-looking variables. In the latter case, there is a simultaneity problem, in that the instrument and the forward-looking variables are simultaneously determined. An implicit instrument rule can be interpreted as an equilibrium condition. As discussed in Svensson [16] and Svensson and Woodford [18], the implementation of an implicit instrument rule is problematic, since any consideration of the practical implementation of an instrument rule leads to the conclusion that a central bank can literally only respond to predetermined variables.[5] We disregard these problems here, and consider (5.1) as just another equilibrium condition added to equations (2.1) and (2.2).

We can write (5.1) in the more general form

$$0 = F_{\tilde{X}tj_t}\tilde{X}_t - F_{\tilde{\imath}tj_t}\tilde{\imath}_t \qquad (0 \leq t \leq T-1), \tag{5.2}$$

where

$$F_{\tilde{\imath}tj_t} \equiv [F_{xtj_t} \;\; -I_{n_i} \;\; 0_{n_i \times n_x}], \tag{5.3}$$

where $\tilde{\imath}_t \equiv (x_t', i_t', \gamma_t')'$. Assume that the policy function shifts permanently to the optimal policy function (2.12) in period T. This is a reasonably general formulation. Since one of the elements

[5] In practice, because of a complex and systematic decision process (Brash [2], Sims [14], Svensson [15]), the information modern central banks respond to is at least a few days or a week old, and most of the information is one or several months old.

of X_t may be unity, (5.1) includes the case of an exogenous time-varying and mode-dependent instrument level for the first T periods, including the case of a constant instrument level.

It follows from section 2 that there exists $\tilde{V}_j$ and $\tilde{w}_j$ $(j = 1, ..., n)$ such that, for $t \geq T$, the intertemporal loss for the modified saddlepoint problem satisfies

$$\tilde{X}_t' \tilde{V}_{j_t} \tilde{X}_t + \tilde{w}_{j_t} \equiv \max_{\gamma_t} \min_{(x_t, i_t)} \{\tilde{L}_t + \mathrm{E}_t \delta (\tilde{X}_{t+1}' \tilde{V}_{j_{t+1}} \tilde{X}_{t+1} + \tilde{w}_{j_{t+1}})\} \qquad (t \geq T)$$

subject to

$$\tilde{X}_{t+1} = \tilde{A}_{j_{t+1}} \tilde{X}_t + \tilde{B}_{j_{t+1}} \tilde{\imath}_t + \tilde{C}_{j_{t+1}} \varepsilon_{t+1} \tag{5.4}$$

and $\tilde{X}_t$ given ($\tilde{X}_t$, $\tilde{L}_t$, $\tilde{\imath}_t$, $\tilde{A}_{j_{t+1}}$, $\tilde{B}_{j_{t+1}}$, and $\tilde{C}_{j_{t+1}}$ are defined as in (2.10) and (2.11)). Recall that this modified intertemporal loss is the intertemporal loss associated with the modified loss function, not the true loss function.

For $t = T-1, T-2, ..., 0$, by the recursive saddlepoint method of Marcet and Marimon [11], we can define $\tilde{V}_{tj_t}$ and $\tilde{w}_{j_t}$ recursively from the saddlepoint problems,

$$\tilde{X}_t' \tilde{V}_{tj_t} \tilde{X}_t + \tilde{w}_{tj_t} \equiv \max_{(\gamma_t, \varphi_t)} \min_{(x_t, i_t)} \left\{ \begin{array}{l} \tilde{L}_t + \varphi_t' \left(-F_{\tilde{X}tj_t} \tilde{X}_t + F_{\tilde{\imath}tj_t} \tilde{\imath}_t \right) \\ + \mathrm{E}_t \delta (\tilde{X}_{t+1}' \tilde{V}_{t+1, j_{t+1}} \tilde{X}_{t+1} + \tilde{w}_{t+1, j_{t+1}}) \end{array} \right\} \qquad (0 \leq t \leq T-1), \tag{5.5}$$

subject to (5.4) and (5.2), where $\tilde{V}_{Tj_t} \equiv \tilde{V}_{j_t}$ and $\tilde{w}_{Tj_t} \equiv \tilde{w}_{j_t}$. Here, φ_t can be interpreted as an n_i-vector of Lagrange multipliers for the n_i equations (5.1). Formally, (5.2) is added to the equations (2.2) and the Lagrange multiplier γ_t is augmented to $(\gamma_t', \varphi_t')'$. Normally, the recursive saddlepoint method would then involve augmenting the Lagrange multiplier Ξ_{t-1} to $(\Xi_{t-1}', \Phi_{t-1}')'$, with the added dynamic equation

$$\Phi_t = \varphi_t.$$

However, the augmented period loss is here

$$\hat{L}_t \equiv \tilde{L}_t + \varphi_t' \left(i_t - F_{\tilde{X}tj_t} \tilde{X}_t - F_{xtj_t} x_t \right). \tag{5.6}$$

Since the analogue of $\mathrm{E}_t H_{t+1} x_{t+1}$, the left side of (5.2), is zero, there is no term including Φ_t' augmented to the period loss. Hence, we do not need to consider Φ_t as an additional predetermined variable here.

The recursive saddlepoint method provides a simple and compact way to incorporating the fact that the equilibrium forward-looking variables x_t and the Lagrange multiplier Ξ_{t-1} will be affected by the constraint (5.1). We have to remember that the resulting value functions are those of the modified period loss, not those of the actual loss.

The solution determines the time- and mode-dependent optimal policy function $\tilde{F}_{tj_t}$,

$$\tilde{\imath}_t \equiv \begin{bmatrix} x_t \\ i_t \\ \gamma_t \end{bmatrix} = \tilde{F}_{tj_t}\tilde{X}_t \equiv \begin{bmatrix} \tilde{F}_{xtj_t} \\ \tilde{F}_{itj_t} \\ \tilde{F}_{\gamma tj_t} \end{bmatrix} \tilde{X}_t \qquad (0 \leq t \leq T-1),$$

where of course i_t in $\tilde{\imath}_t$ satisfies (5.1). The interesting part of the solution is

$$x_t = \tilde{F}_{xtj_t}\tilde{X}_t, \tag{5.7}$$

and $\tilde{F}_{xtj_t}$ will satisfy

$$\tilde{F}_{itj_t} \equiv F_{\tilde{X}tj_t} + F_{xtj_t}\tilde{F}_{xtj_t}.$$

There is also a solution for φ_t, $\varphi_t = \tilde{F}_{\varphi tj_t}\tilde{X}_t$, but that solution is not needed for the intertemporal loss and the dynamics. It follows that the dynamics of $\tilde{X}_t$ satisfies

$$\begin{aligned} \tilde{X}_{t+1} &= M_{tj_tj_{t+1}}\tilde{X}_t + \tilde{C}_{j_{t+1}}\varepsilon_{t+1} \qquad (0 \leq t \leq T-1), \\ \tilde{X}_{t+1} &= M_{j_tj_{t+1}}\tilde{X}_t + \tilde{C}_{j_{t+1}}\varepsilon_{t+1} \qquad (t \geq T) \end{aligned}$$

where

$$\begin{aligned} M_{tj_tj_{t+1}} &\equiv \tilde{A}_{j_{t+1}} + \tilde{B}_{j_{t+1}}\tilde{F}_{tj_t} \qquad (0 \leq t \leq T-1), \\ M_{j_tj_{t+1}} &\equiv \tilde{A}_{j_{t+1}} + \tilde{B}_{j_{t+1}}\tilde{F}_{j_t} \qquad (t \geq T). \end{aligned}$$

The intertemporal loss in period 0 for the modified period loss function (5.6) will be given by

$$\tilde{X}_0'\tilde{V}_{0j_0}\tilde{X}_0 + \tilde{w}_{0j_0}.$$

However, this is not the intertemporal loss in period 0 for the original period loss function, (2.4). In order to find that, note that the intertemporal loss for the optimal policy for $t \geq T$ will be given by

$$\tilde{X}_t'V_{j_t}\tilde{X}_t + w_{j_t},$$

where the matrix V_j will satisfy the Lyapunov function (2.15) and the constant w_j will satisfy (2.16).

For $t = T-1, T-2, ..., 0$, we can define V_{tj} and w_{tj} recursively from equations

$$\bar{W}_{tj} \equiv \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{\tilde{F}_{txj}} \\ \multicolumn{2}{c}{\tilde{F}_{tij}} \end{bmatrix}' W_j \begin{bmatrix} I & 0 \\ \multicolumn{2}{c}{\tilde{F}_{txj}} \\ \multicolumn{2}{c}{\tilde{F}_{tij}} \end{bmatrix}.$$

$$V_{tj} \equiv \bar{W}_{tj} + \delta \sum_k P_{jk} M'_{tjk} V_{t+1,k} M_{tjk},$$

$$w_{tj_t} \equiv \delta \sum_k P_{jk}[\operatorname{tr}(V_{t+1,k}\sigma^2 \tilde{C}_k \tilde{C}'_k) + w_{t+1,k}],$$

where $V_{Tj} \equiv V_j$ and $w_{Tj} \equiv w_j$.

Then, the intertemporal loss in period 0 for the original period loss function (5.6) is

$$\tilde{X}'_0 V_{0j_0} \tilde{X}_0 + w_{0j_0}.$$

This corresponds to the loss under commitment in a timeless perspective when the instrument is restricted to fulfill (5.1) and shifts to optimal policy in period T. That is, when the restriction (5.1) is removed in period T and optimal policy is feasible, the commitment is not from scratch in period T (in which case Ξ_{T-1} would equal zero) but takes into account the previous Lagrange multiplier Ξ_{T-1}. In principle, this formulation also allows us to consider nonzero Ξ_{-1} in period 0.

The above recursive saddlepoint method also works for the backward-looking case, in which case

$$\tilde{L}_t \equiv L_t$$

and there are no variables γ_t, x_t, and Ξ_{t-1} (equivalently, they are identically equal to zero). Then the intertemporal loss for the saddlepoint problem is equal to the intertemporal loss for the original problem.

Details about the computation of $\tilde{F}_{tj_t}$ are provided in appendix D.

5.1 Optimization

Let $F_t \equiv \{F_{\tilde{X}tj_t}, F_{xtj_t}\}_{j_t=1}^n$ for $0 \leq t \leq T-1$, and let $F \equiv \{F_t\}_{t=0}^{T-1}$ denote the given time- and mode-dependent policy functions for $0 \leq t \leq T-1$. We may assume that there is a feasible set $\mathcal{F}$ of such policy functions, so $F \in \mathcal{F}$. Then we can, in principle, consider choosing the policy functions optimally according to

$$\min_{F \in \mathcal{F}} \tilde{X}'_0 V_{0j_0}(F) \tilde{X}_0 + w_{0j_0}(F), \tag{5.8}$$

where the notation emphasizes that that V_{0j_t} and w_{0j_t} will depend on F. With the policy problem formulated this way, the optimal F would depend on $\tilde{X}_0$ (including Ξ_{-1}) and j_0 as well as the covariance matrix $\sigma^2 \tilde{C}_k \tilde{C}'_k$ of the shocks $\tilde{C}_k \varepsilon_{t+1}$ to $\tilde{X}_{t+1}$ in mode $j_{t+1} = k$ $(k = 1, ..., n)$. If the class of time- and mode-dependent policy functions is sufficiently big, it would include the optimal policy function (2.12). If we would add $\frac{1}{\delta}\Xi_{-1} H_{j_0} x_0$ to the period loss function in period 0, the optimal policy function would then be the solution to (5.8).

Note that, if $\mathcal{F}$ is such that $F_{xtj_t} \neq 0$, the optimal F is generally not unique. The reason is that for (5.7), if

$$i_t = F_{\tilde{X}tj_t}\tilde{X}_t + F_{xtj_t}x_t$$

is a solution, so is

$$i_t = F_{\tilde{X}tj_t}\tilde{X}_t + F_{xtj_t}x_t + \theta'(x_t - \tilde{F}_{xj_t}\tilde{X}_t) = (F_{\tilde{X}tj_t} - \theta'\tilde{F}_{xj_t})\tilde{X}_t + (F_{xtj_t} + \theta')x_t$$

for any n_x-vector θ.

6 Arbitrary time-invariant instrument rules and optimal restricted instrument rules

In this section we derive the dynamics of the system, including the distribution of forecasts of relevant future variables, for arbitrary time-invariant instrument rules. We also specify the optimization problem for time-invariant instrument rules in a given class of instrument rules.

Consider an arbitrary time-invariant instrument rule,

$$i_t = F_{\tilde{X}j_t}\tilde{X}_t + F_{xj_t}x_t \qquad (j_t = 1, ..., n), \tag{6.1}$$

combined with (2.1) and (2.2). We can consider this as a special case of the time-invariant instrument rules in section 5, if we let $F_{\tilde{X}tj_t} = F_{\tilde{X}j_t}$ and $F_{xtj_t} = F_{xj_t}$ and apply the algorithm of that section by iterating from $t = T > t_0$ to $t = t_0$ but instead of stopping at $t_0 = 0$ letting $t_0 \to -\infty$. In practice, the iteration would stop when $\tilde{F}_{tj_t}$ and $\tilde{V}_{tj_t}$ have converged to $\tilde{F}_{j_t}$ and $\tilde{V}_{j_t}$. Partitioning $\tilde{F}_{j_t}$ conformably with x_t, i_t, and γ_t, we have

$$\begin{aligned}
x_t &= \tilde{F}_{xj_t}\tilde{X}_t, \\
i_t &= F_{\tilde{X}j_t}\tilde{X}_t + F_{xj_t}\tilde{F}_{xj_t}X_t \equiv \tilde{F}_{ij_t}X_t, \\
\tilde{X}_{t+1} &= M_{j_tj_{t+1}}\tilde{X}_t + \tilde{C}_{j_{t+1}}\varepsilon_{t+1} \qquad (j_t = 1, ..., n).
\end{aligned}$$

This gives rise to a probability distribution of $\tilde{X}_{t+\tau}$, $x_{t+\tau}$, and $i_{t+\tau}$ $(\tau \geq 0)$ conditional on $\tilde{X}_t$ and j_t.

This solution will be associated with a value function for the original period loss function,

$$\tilde{X}_t'V_{j_t}\tilde{X}_t + w_{j_t}.$$

6.1 Optimization

For a given restricted class $\mathcal{F}$ of instrument rules, we can consider the optimal restricted (time-invariant) instrument rule $\hat{F}$, the instrument rule in $\mathcal{F}$ that minimizes an intertemporal loss function.

This intertemporal loss function could be the conditional loss in a given period, say period 0,

$$\hat{F} \equiv \arg\min \tilde{X}_0' V_{j_0}(F) \tilde{X}_0 + w_{j_0}(F),$$

where the notation takes into account that $V_{j_0}(F)$ and $w_{j_0}(F)$ depend on $F \in \mathcal{F}$. This would make the optimal restricted time-invariant instrument rule depend on $\tilde{X}_0$, j_0, and the covariance matrices $\sigma^2 \tilde{C}_j \tilde{C}_j'$ of the shocks $\tilde{C}_j \varepsilon_{t+1}$ to $\tilde{X}_{t+1}$ in mode $j = 1, ..., n$.

The intertemporal loss function could also be the unconditional mean of the period loss function, $\mathrm{E}[L_t]$,

$$\hat{F} = \arg\min_{F \in \mathcal{F}} \mathrm{E}[L_t].$$

Note that

$$\mathrm{E}[L_t] = (1-\delta)\{\mathrm{E}[\tilde{X}_t' V_{j_t}(F) \tilde{X}_t + w_{j_t}(F)\} = (1-\delta)\{\mathrm{tr}[V_{j_t} \mathrm{E}(\tilde{X}_t \tilde{X}_t')] + w_{j_t}\}.$$

Furthermore, the unconditional and conditional loss are approximately the same when the unconditional loss is scaled by $1-\delta$ and δ is close to one,

$$\lim_{\delta \to 1-} \mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^\tau L_{t+\tau} = \mathrm{E}[L_t] = \lim_{\delta \to 1-} (1-\delta)\mathrm{E}[w_{j_t}] = \mathrm{E}[\mathrm{tr}(V_{j_t} \sigma^2 \tilde{C}_{j_t} \tilde{C}_{j_t}')]$$

7 Policy conditional only on the distribution of modes

In this section, we show how the optimal policy and value functions can be expressed as a function of the probability distribution of model modes, when these modes are not observed.

Assume that central bank knows the distribution of modes in period t, p_t, but does not observe the actual mode. Conditional on p_t in period t, the distribution of the modes in period $t+\tau$ is given by

$$p_{t+\tau} = (P')^\tau p_t \qquad (\tau \geq 0).$$

We examine the optimal policy function and the value function under the simplification that there are no forward-looking variables, so the model is

$$X_{t+1} = A_{j_{t+1}} X_t + B_{j_{t+1}} i_t + C_{j_{t+1}} \varepsilon_{t+1}.$$

Then the recursive saddlepoint method need to be applied. We also assume that the period loss function,

$$L_t = \left[\begin{array}{c} X_t \\ i_t \end{array} \right]' W \left[\begin{array}{c} X_t \\ i_t \end{array} \right],$$

is independent of the mode.

The optimal policy function and the value function can then be written

$$i_t = F(p_t) X_t,$$

$$X_t' V(p_t) X_t + w(p_t).$$

Appendix E shows how the functions $F(p_t)$, $V(p_t)$, and $w(p_t)$ can be computed by modifying the iterations specified in appendix B. Computing the functions $F(p_t)$ and $V(p_t)$ for all feasible values of p_t requires standard function approximation methods. Computing the functions for a particular value $p_t = \tilde{p}_t$ is straightforward.

Consider the degenerate distributions, $p_t = e_j$ where e_j is the distribution where $p_j = 1$, $p_k = 0$ $(k \neq j)$. That is, e_j corresponds to the case when the mode j is observed in period t. Note that $V(e_j) \neq V_j$ and $F(e_j) \neq F_j$, where V_j and F_j $(j = 1, ..., n)$ denote the value function and optimal policy function matrices for the case when the modes are observed in each period. The reason is that even if $p_t = e_j$ and the mode is observed in this period, the distribution of the modes in the next period will be $p_{t+1} = P' e_j = (P_{j1}, P_{j2}, ..., P_{jn})'$ and the modes will not be observed in the next period. In contrast, V_j and F_j follow under the assumption that the modes are observed in this period and every future period.

8 Conclusions

[To be added.]

Appendix

A Incorporating central-bank judgment

In order to incorporate (additive) central-bank judgment as in Svensson [16], consider the model

$$X_{t+1} = A_{11,t+1}X_t + A_{12,t+1}x_t + B_{1,t+1}i_t + C_{t+1}z_{t+1}, \quad \text{(A.1)}$$

$$\mathrm{E}_t H_{t+1}x_{t+1} = A_{21,t}X_t + A_{22,t}x_t + B_{2,t}i_t, \quad \text{(A.2)}$$

where z_t, the (additive) *deviation*, is a an exogenous n_z-vector stochastic process. Assume that z_t satisfies

$$z_{t+1} = \varepsilon_{t+1} + \sum_{j=1}^{T} \varepsilon_{t+1,t+1-j},$$

where $(\varepsilon_t', \varepsilon^{t\prime})' \equiv (\varepsilon_t', \varepsilon_{t+1,t}', ..., \varepsilon_{t+T,t}')'$ is a zero-mean i.i.d. random $(T+1)n_z$-vector realized in the beginning of period t and called the innovation in period t. For $T = 0$, $z_{t+1} = \varepsilon_{t+1}$ is a simple i.i.d. disturbance. For $T > 0$, the deviation is a version of a moving-average process.

The dynamics of the deviation can be written

$$\begin{bmatrix} z_{t+1} \\ z^{t+1} \end{bmatrix} = A_z \begin{bmatrix} z_t \\ z^t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon^{t+1} \end{bmatrix},$$

where $z^t \equiv (\mathrm{E}_t z_{t+1}', \mathrm{E}_t z_{t+2}', ..., \mathrm{E}_t z_{t+T})'$ can be interpreted as the central bank's (additive) *judgment* in period t and the $(T+1)n_z \times (T+1)n_z$ matrix A_z is defined as

$$A_z \equiv \begin{bmatrix} 0_{n_z \times n_z} & I_{n_z} & 0_{n_z \times (T-1)n_z} \\ 0_{(T-1)n_z \times n_z} & 0_{(T-1)n_z \times n_z} & I_{(T-1)n_z} \\ 0_{n_z \times n_z} & 0_{n_z \times n_z} & 0_{n_z \times (T-1)n_z} \end{bmatrix} \equiv \begin{bmatrix} 0 & A_{z21} \\ 0 & A_{z22} \end{bmatrix};$$

in the second identity A_z is partitioned conformably with z_t and z^t. Hence z^t is the central bank's mean projection of future deviations, and ε^t can be interpreted as the new information the central bank receives in period t about those future deviations.[6]

It follows that the model can be written in the mode-space form (2.1) and (2.2) as

$$\begin{bmatrix} X_{t+1} \\ z_{t+1} \\ z^{t+1} \end{bmatrix} = \hat{A}_{11,t+1} \begin{bmatrix} X_t \\ z_t \\ z^t \end{bmatrix} + \hat{A}_{12,t+1}x_t + \hat{B}_{1,t+1}i_t + \hat{C}_{t+1} \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon^{t+1} \end{bmatrix},$$

$$\mathrm{E}_t H_{t+1}x_{t+1} = \hat{A}_{21,t} \begin{bmatrix} X_t \\ z_t \\ z^t \end{bmatrix} + A_{22,t}x_t + B_{2,t}i_t,$$

[6] The graphs in Svensson [16] can be seen as impulse responses to ε^t.

where

$$\hat{A}_{11,t+1} \equiv \begin{bmatrix} A_{11,t+1} & 0 & C_{t+1}A_{z21} \\ 0 & 0 & A_{z21} \\ 0 & 0 & A_{z22} \end{bmatrix}, \quad \hat{B}_{1,t+1} \equiv \begin{bmatrix} B_{1,t+1} \\ 0 \\ 0 \end{bmatrix}, \quad \hat{C}_{t+1} \equiv \begin{bmatrix} 0 & C_{t+1} \\ I_{n_z} & 0 \\ 0 & I_{n_z} \end{bmatrix},$$

$$\hat{A}_{11,t+1} \equiv \begin{bmatrix} A_{21,t+1} \\ 0 \\ 0 \end{bmatrix},$$

and the new predetermined variables are $(X_t', z_t', z^{t\prime})'$.

B An algorithm for the value function and optimal policy function

Consider the modified saddlepoint problem of (2.6) subject to (2.7), (2.8), and $\tilde{X}_t$ given. Let us use the notation $Z_t = Z_{j_t}$ for any matrix Z that is a function of the mode j_t, and let the matrix $\tilde{W}_t = \tilde{W}_{j_t}$ be partitioned conformably with $\tilde{X}_t$ and $\tilde{\imath}_t$ as

$$\tilde{W}_t \equiv \begin{bmatrix} Q_t & N_t \\ N_t' & R_t \end{bmatrix}.$$

We use that the value function for the modified problem will be quadratic and can be written

$$\tilde{X}_t'\tilde{V}_t\tilde{X}_t + \tilde{w}_t,$$

where $\tilde{V}_t$ is a matrix and $\tilde{w}_t$ a scalar. It will fulfill the Bellman equation

$$\tilde{X}_t'\tilde{V}_t\tilde{X}_t + \tilde{w}_t = \max_{\gamma_t} \min_{(x_t,i_t)} \left\{ \tilde{X}_t'Q_t\tilde{X}_t + 2\tilde{X}_t'N_t\tilde{\imath}_t + \tilde{\imath}_t'R_t\tilde{\imath}_t + \mathrm{E}_t(\tilde{X}_{t+1}\tilde{V}_{t+1}\tilde{X}_{t+1} + \tilde{w}_{t+1}) \right\},$$

where $\tilde{X}_{t+1}$ is given by (2.8) and E_t refers to the expectations conditional on $\tilde{X}_t$ and j_t.

The first-order condition with respect to $\tilde{\imath}_t$ is

$$X_t'N_t + \tilde{\imath}_t'R_t + \tilde{X}_t'\mathrm{E}_t\tilde{A}_{t+1}'\tilde{V}_{t+1}\tilde{B}_{t+1} + \tilde{\imath}_t'\mathrm{E}_t\tilde{B}_{t+1}'\tilde{V}_{t+1}\tilde{B}_{t+1} = 0,$$

which can be written

$$J_t\tilde{\imath}_t + K_t\tilde{X}_t = 0,$$

where

$$J_t \equiv R_t + \mathrm{E}_tB_{t+1}'\tilde{V}_{t+1}\tilde{B}_{t+1}, \tag{B.1}$$

$$K_t \equiv \mathrm{E}_t\tilde{B}_{t+1}'\tilde{V}_{t+1}\tilde{A}_{t+1} + N_t'. \tag{B.2}$$

This leads to the optimal policy function

$$i_t = F_t\tilde{X}_t, \tag{B.3}$$

where

$$F_t \equiv -J_t^{-1}K_t. \tag{B.4}$$

Furthermore, the value function satisfies

$$\tilde{X}_t'\tilde{V}_t\tilde{X}_t \equiv \tilde{X}_t'Q_t\tilde{X}_t + 2\tilde{X}_t'N_tF_t\tilde{X}_t + \tilde{X}_t'F_t'R_tF_t\tilde{X}_t + \tilde{X}_t'\mathrm{E}_t[(\tilde{A}_{t+1}' + F_t'\tilde{B}_{t+1}')\tilde{V}_{t+1}(\tilde{A}_{t+1} + \tilde{B}_{t+1}F_t)]\tilde{X}_t.$$

This implies

$$\tilde{V}_t = Q_t + N_tF_t + F_t'N_t' + F_t'R_tF_t + \mathrm{E}_t[(\tilde{A}_{t+1}' + F_t'\tilde{B}_{t+1}')\tilde{V}_{t+1}(\tilde{A}_{t+1} + \tilde{B}_{t+1}F_t)],$$

which can be simplified to the Riccati equation

$$\tilde{V}_t = Q_t + \mathrm{E}_t\tilde{A}_{t+1}'\tilde{V}_{t+1}\tilde{A}_{t+1} - K_t'J_t^{-1}K_t. \tag{B.5}$$

Equations (B.1), (B.2), and (B.5) show how $V_{t+1} = V_{j_{t+1}}$ for $j_{t+1} = 1, ..., n$ is mapped into $V_t = V_{j_t}$ for $j_t = 1, ..., n$.

Iteration backwards of (B.4) and (B.5) from any constant positive semidefinite matrix $\tilde{V}$ should converge to stationary matrices functions F_j and $\tilde{V}_j$ $(j = 1, ..., n)$, where V_j satisfies the Riccati equation (B.5) with (B.1) and (B.2).

Taking account of the finite number of modes, we have

$$\begin{aligned}
F_j &\equiv -J_j^{-1}K_j \\
J_j &\equiv R_j + \sum_{k=1}^{n} \tilde{B}_k'\tilde{V}_k\tilde{B}_kP_{jk} \\
K_j &\equiv \sum_{k=1}^{n} \tilde{B}_k'\tilde{V}_k\tilde{A}_kP_{jk} + N_j', \\
\tilde{V}_j &= Q_j + \sum_{k=1}^{n} \tilde{A}_k'\tilde{V}_k\tilde{A}_kP_{jk} - K_j'J_j^{-1}K_j \qquad (j = 1, ..., n),
\end{aligned} \tag{B.6}$$

where P_{jk} is the transition probability from $j_t = j$ to $j_{t+1} = k$.

The scalars $\tilde{w}_j$ will fulfill the equations

$$\tilde{w}_j = \delta \sum_k P_{jk}[\mathrm{tr}(\tilde{V}_k\sigma^2\tilde{C}_k\tilde{C}_k') + \tilde{w}_k].$$

Thus determining the optimal policy function (B.3) reduces to solving a system of coupled algebraic Riccati equations (B.6). In order to solve this system numerically, we adapt the algorithm of do Val, Geromel, and Costa [6]. In a very similar problem, they show how the coupled Riccati equations can be uncoupled for numerical solution.[7]

[7] In their problem, the matrices A and B next period are known in the current period, so the averaging in the Riccati equation is only over the V_j matrices.

The algorithm consists of the following steps:

1. Define $\hat{A}_j = \sqrt{P_{jj}}\tilde{A}_j$, $\hat{B}_j = \sqrt{P_{jj}}\tilde{B}_j$ and initialize $\tilde{V}_j^0 = 0$, $j = 1, \ldots, n$.

2. Then at each iteration $l = 0, 1, \ldots$, for each j define:

$$\begin{aligned}
\hat{Q}_j &= Q_j + \sum_{k \neq j} \tilde{A}_k' \tilde{V}_k^l \tilde{A}_k P_{jk} \\
\hat{R}_j &= R_j + \sum_{k \neq j} \tilde{B}_k' \tilde{V}_k^l \tilde{B}_k P_{jk} \\
\hat{N}_j &= N_j + \sum_{k \neq j} \tilde{A}_k' \tilde{V}_k^l \tilde{B}_k P_{jk}.
\end{aligned}$$

   Then for each j solve the standard Riccati equation for the problem with matrices $(\hat{A}_j, \hat{B}_j, \hat{Q}_j, \hat{R}_j, \hat{N}_j)$. Note that these are uncoupled since $\tilde{V}_k^l$ is known. Call the solution $\tilde{V}_j^{l+1}$.

3. Check $\sum_{j=1}^n \|\tilde{V}_j^{l+1} - \tilde{V}_j^l\|$. If this is lower then a tolerance, stop. Otherwise, return to step 2.

do Val, Geromel, and Costa [6] show that the sequence of matrices $\tilde{V}_j^l$ converges to the solution of (B.6) as $l \to \infty$. In order to understand the algorithm, recall that, in the standard linear-quadratic regulator (LQR) problem (Anderson, Hansen, McGrattan, and Sargent [1] and Ljungqvist and Sargent [10]), we have

$$\begin{aligned}
F &\equiv -J^{-1}K \\
J &\equiv R + B'VB \\
K &\equiv B'VA + N', \\
V &= Q + A'VA - K'J^{-1}K.
\end{aligned}$$

If we can redefine the matrices so the equations conform to the standard case, we can use the standard algorithm for the LQR problem to find F_j and V_j. The above definitions indeed allow us to write

$$\begin{aligned}
F_j &\equiv -J_j^{-1}K_j, \\
J_j &\equiv \hat{R}_j + \hat{B}_j'\tilde{V}_j\hat{B}_j, \\
K_j &\equiv \hat{B}_j'\tilde{V}_j\hat{A}_j + \hat{N}_j', \\
\tilde{V}_j &= \hat{Q}_j + \hat{A}_j'\tilde{V}_j\hat{A}_j - K_j'J_j^{-1}K_j \qquad (j = 1, ..., n),
\end{aligned}$$

so we can indeed use the standard algorithm.

Note that the above algorithm is easily modified to solve the Lyapunov equation (2.15) for the matrix V_j for the true value function of the original problem.

C Alternative models with different predetermined and forward-looking model

Our MJLQ framework allows us to consider situations when the modes $j = 1, ..., n$ correspond to alternative structural models, including not only when some coefficients are zero or nonzero but also when a variable is predetermined in one model and forward-looking in another. This allows us include optimal policy when it is known what structural model is true in the current period but there is uncertainty about the true structural model in the future.[8]

In order to see this, consider a particular simple example, when there are two modes, $j = 1, 2$, with transition matrix $P = [P_{jk}]$, $j, k = 1, 2$. Let $j = 1$ corresponds to a model with an acceleration Phillips curve (the AP model),

$$\pi_{t+1} = \pi_t + \alpha y_t + \varepsilon_{1,t+1},$$

and let $j = 2$ corresponds to a New Keynesian Phillips curve (the NK model),

$$\mathrm{E}_t \pi_{t+1} = \pi_t - \kappa y_t - \varepsilon_{2,t},$$

where ε_{1t} and ε_{2t} are i.i.d. with zero means. Thus, inflation, π_{t+1} is predetermined in AP model and forward-looking in the NK model. Regard the output gap, y_t, as the control variable, for simplicity.

Let π_t denote *actual* inflation in period t, and introduce the two variables π_{1t} and π_{2t}, where π_{1t} is predetermined and denotes inflation in the AP model (AP inflation) and π_{2t} is forward-looking and denotes inflation in the NK model (NK inflation). Actual actual inflation then satisfies

$$\pi_t = \theta_t \pi_{1t} + (1 - \theta_t)\pi_{2t},$$

where $\theta_t = 1$ in mode 1 and $\theta_t = 0$ in mode 2. We thus have

$$\begin{aligned} \pi_{1,t+1} &= \pi_t + \alpha y_t + \varepsilon_{1,t+1}, \\ \mathrm{E}_t \pi_{t+1} &= \pi_{2t} - \kappa y_t - \varepsilon_{2t}, \end{aligned} \tag{C.1}$$

where we assume that, in the AP model, current *actual* inflation affects future AP inflation and, in the NK model, the expected future *actual* inflation affects current NK inflation.

We want to write this model as (2.1) and (2.2) by suitable definitions of X_t, x_t, i_t, and ε_t, and the matrices. The trick is to treat actual inflation, π_t, as a nonpredetermined variable even though

[8] If the current model is not observed, we would have to include Bayesian learning of the subjective probability distribution over models and encounter problems of experimentation versus "adaptive" loss minimization [give reference(s)].

this is not the case when the AP model is true. This works, because an additional predetermined variable identical to an existing predetermined variable can always be introduced as a trivial non-predetermined variable by adding an equation in the block of equations for the forward-looking variables. Suppose that the new variable, y_t, is identical to an existing predetermined variable, X_{1t}, say. Then we can just add the equation

$$0 = X_{1t} - y_t,$$

to that block, where the left side has zero instead of a linear combination of expected future forward-looking variables. Generally, a new variable that is a linear combination of current predetermined and forward-looking variables can always be introduced as a new forward-looking variable in this way.

Observe that

$$\begin{aligned} \mathrm{E}_t \pi_{t+1} &= \mathrm{E}_t[\theta_{t+1}\pi_{1,t+1} + (1-\theta_{t+1})\pi_{2,t+1}] \\ &= \mathrm{E}_t \theta_{t+1}(\pi_t + \alpha y_t) + \mathrm{E}_t(1-\theta_{t+1})\pi_{2,t+1} \end{aligned}$$

and use this to substitute for $\mathrm{E}_t\pi_{t+1}$ in (C.1). Let $X_t \equiv (\pi_{1t}, \varepsilon_{2t})'$, $x_t \equiv (\pi_{2t}, \pi_t)'$, and $i_t \equiv y_t$. Then we can write the model in the form (2.1) and (2.2) as

$$X_{t+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_t + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} i_t + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}$$

$$\mathrm{E}_t \begin{bmatrix} 1-\theta_{t+1} & 0 \\ 0 & 0 \end{bmatrix} x_{t+1} = \begin{bmatrix} 0 & -1 \\ \theta_t & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 & 1-\mathrm{E}_t\theta_{t+1} \\ 1-\theta_t & -1 \end{bmatrix} x_t + \begin{bmatrix} -\kappa - \alpha \mathrm{E}_t \theta_{t+1} \\ 0 \end{bmatrix} i_t.$$

D Details for arbitrary time-varying instrument rules

For $t = 0, ..., T-1$, introduce the new $(n_{\tilde{\imath}} + n_i)$-vector of instruments,

$$\hat{\imath}_t \equiv \begin{bmatrix} \tilde{\imath}_t \\ \varphi_t \end{bmatrix},$$

and write the model

$$\tilde{X}_{t+1} = \tilde{A}_{j_{t+1}} \tilde{X}_t + \hat{B}_{j_{t+1}} \hat{\imath}_t + \tilde{C}_{j_{t+1}} \varepsilon_{t+1},$$

where the new $n_{\tilde{X}} \times (n_{\tilde{\imath}} + n_i)$ matrix $\hat{B}_{j_{t+1}}$ satisfies

$$\hat{B}_{j_{t+1}} \equiv \begin{bmatrix} \tilde{B}_{j_{t+1}} & 0_{n_{\tilde{X}} \times n_i} \end{bmatrix}.$$

Partition the $(n_{\tilde{X}} + n_{\tilde{\imath}}) \times (n_{\tilde{X}} + n_{\tilde{\imath}})$ matrix $\tilde{W}_{j_t}$ conformably with $\tilde{X}_t$ and $\tilde{\imath}_t$ as

$$\tilde{W}_{j_t} = \begin{bmatrix} Q_{j_t} & N_{j_t} \\ N'_{j_t} & R_{j_t} \end{bmatrix}.$$

Furthermore, write the augmented period loss as

$$\hat{L}_t \equiv \begin{bmatrix} \tilde{X}_t \\ \hat{\imath}_t \end{bmatrix}' \begin{bmatrix} Q_{j_t} & \hat{N}_{j_t} \\ \hat{N}'_{j_t} & \hat{R}_{j_t} \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ \hat{\imath}_t \end{bmatrix},$$

where the new $n_{\tilde{X}} \times (n_{\tilde{\imath}} + n_x)$ and $(n_{\tilde{\imath}} + n_x) \times (n_{\tilde{\imath}} + n_x)$ matrices $\hat{N}_{j_t}$ and $\hat{R}_{j_t}$ satisfy, respectively,

$$\hat{N}_{tj_t} \equiv \begin{bmatrix} N_{j_t} & -F'_{\tilde{X}tj_t}/2 \end{bmatrix}, \qquad \hat{R}_{tj_t} \equiv \begin{bmatrix} R_{j_t} & -F'_{\tilde{\imath}tj_t}/2 \\ -F_{\tilde{\imath}tj_t}/2 & 0 \end{bmatrix}.$$

Then, the first-order condition for an optimum of the Bellman equation will, in the standard way, result in a time- and mode-dependent optimal policy function

$$\hat{\imath}_t = \hat{F}_{tj_t}\tilde{X}_t \qquad (0 \leq t \leq T-1,\ 0 \leq j_t \leq n),$$

which is defined in a compact way as

$$\hat{F}_{tj_t} \equiv -\,J^{-1}_{tj_t}K_{tj_t},$$

where J_{tj_t} and K_{tj_t} are defined recursively from $\tilde{V}_{t+1,j_t}$ as

$$\begin{aligned} J_{tj_t} &\equiv \hat{R}_{tj_t} + \mathrm{E}_t\hat{B}'_{j_{t+1}}\tilde{V}_{t+1,j_{t+1}}\hat{B}_{j_{t+1}} = \hat{R}_{tj_t} + \sum_k \hat{B}'_k\tilde{V}_{t+1,k}\hat{B}_kP_{j_tk}, \\ K_{tj_t} &\equiv \mathrm{E}_t\hat{B}'_{j_{t+1}}\tilde{V}_{t+1,j_{t+1}}\tilde{A}_{j_{t+1}} + \hat{N}'_{j_t} = \sum_k \hat{B}'_k\tilde{V}_{t+1,k}\tilde{A}_k + \hat{N}'_{tj_t}. \end{aligned}$$

Substitution of this optimal policy function in the Bellman equation results in the recursive equation for $\tilde{V}_{tj_t}$,

$$\tilde{V}_{tj_t} = Q_{j_t} + \mathrm{E}_t\tilde{A}'_{j_{t+1}}\tilde{V}_{t+1,j_{t+1}}\tilde{A}_{j_{t+1}} - K'_{tj_t}J^{-1}_{tj_t}K_{tj_t} = Q_{j_t} + \sum_k \tilde{A}'_k\tilde{V}_{t+1,k}\tilde{A}_k - K'_{tj_t}J^{-1}_{tj_t}K_{tj_t}.$$

Finally, the optimal policy function $\tilde{F}_{tj_t}$ for $t = 0, ..., T-1$ can be identified by partitioning $\hat{F}_{tj_t}$ conformably with $\tilde{\imath}_t$ and φ_t,

$$\hat{F}_{tj_t} \equiv \begin{bmatrix} \tilde{F}_{tj_t} \\ F_{\varphi tj_t} \end{bmatrix}.$$

E Details when modes are not observable

We adapt the iterative process we have used in appendix B to determine F_j and V_j in the present case. We assume that beginning sometime far into the future, the modes can be observed. Once the modes are observed, we have the mode-dependent value-function matrices, V_j $(j = 1, ..., n)$ determined in appendix B. Consider the period before the modes can be observed, let $p = (p_1, ..., p_n)'$ denote an arbitrary distribution in that mode, and let $V^0(p)$ be the matrix of the value function in that period. Think of this as iteration $l = 0$. The matrix function $V^0(p)$ will be given by

$$\begin{aligned} J^0(p) &\equiv R + \sum_k \sum_j p_j P_{jk} B_k' V_k B_k, \\ K^0(p) &\equiv \sum_k \sum_j p_j P_{jk} B_k' V_k A_k + N', \\ V^0(p) &= Q + \sum_k \sum_j p_j P_{jk} A_k' V_k A_k - K^0(p)' J^0(p)^{-1} K^0(p), \end{aligned}$$

where the matrix W is partitioned conformably with X_t and i_t as

$$W_j \equiv \left[\begin{array}{cc} Q & N \\ N' & R \end{array} \right].$$

Given this, consider the iteration for $l = 1, 2, ...,$

$$\begin{aligned} J^l(p) &\equiv R + \sum_k \sum_j p_j P_{jk} B_k' V^{l-1}(P'p) B_k \\ K^l(p) &\equiv \sum_k \sum_j p_j P_{jk} B_k' V^{l-1}(P'p) A_k + N' \\ V^l(p) &= Q + \sum_k \sum_j p_j P_{jk} A_k' V^{l-1}(P'p) A_k - K^l(p)' J^l(p)^{-1} K^l(p). \end{aligned}$$

Continue these iterations until $V^l(p)$ has converged, which gives $J(p)$, $K(p)$, and $V(p)$. The policy function is then given by

$$F(p) = -J(p)^{-1} K(p).$$

Note that $V^{l-1}(P'p)$ in the above iteration takes into account that, if the distribution is p this period, it is $P'p$ next period. Also, in the sums above, $V^{l-1}(P'p)$ does not depend on the mode k next period (except for $l = 0$). Furthermore, the current distribution matters only because of the information about the future distribution it conveys. Finally, consider $p_t = e_j$, where e_j is the distribution where $p_j = 1$, $p_k = 0$ $(k \neq j)$. That is, e_j corresponds to the case when the mode j is observed in period t. Note that it does not follow that $V(e_j) = V_j$. This equality would follow, if the mode were observed in each period in the future, but in the above case, even if the mode is by chance observed in the current period, it is not observed in the future period. The distribution in period $t+1$ is then $p_{t+1} = P'e_j = (P_{j1}, P_{j2}, ..., P_{jn})'$.

Note that from the above follows that, in a particular period t, we can always find $V(p_t)$ and $F(p_t)$ for a given $p_t = \tilde{p}$ with the following algorithm: Let $\tau = 1, 2, ..., T-1$ refer to periods $t+\tau$ ahead for a given $T \geq 1$, and define

$$\tilde{p}_\tau \equiv \left(P'\right)^\tau \tilde{p} \qquad (\tau = 1, ..., T-1).$$

Hence, $\tilde{p}_\tau$ denotes the probability distribution of the modes $j_{t+\tau}$ in period $t+\tau$ conditional on the current distribution $p_t = \tilde{p}$. Assume for convenience that the modes become observable beginning in period $t+T$ and define

$$J^{T-1} \equiv R + \sum_k \sum_j \tilde{p}_{T-1,j} P_{jk} B_k' V_k B_k, \tag{E.1}$$

$$K^{T-1} \equiv \sum_k \sum_j \tilde{p}_{T-1,j} P_{jk} B_k' V_k A_k + N', \tag{E.2}$$

$$V^{T-1} = Q + \sum_k \sum_j \tilde{p}_{T-1,j} P_{jk} A_k' V_k A_k - (K^{T-1})'(J^{T-1})^{-1} K^{T-1}. \tag{E.3}$$

Given this, consider the iteration for $l = T-2, ..., 0$,

$$J^l \equiv R + \sum_k \sum_j \tilde{p}_{lj} P_{jk} B_k' V^{l+1} B_k, \tag{E.4}$$

$$K^l \equiv \sum_k \sum_j \tilde{p}_{lj} P_{jk} B_k' V^{l+1} A_k + N', \tag{E.5}$$

$$V^l = Q + \sum_k \sum_j \tilde{p}_{lj} P_{jk} A_k' V^{l+1} A_k - (K^l)'(J^l)^{-1} K^l. \tag{E.6}$$

Then,

$$\begin{aligned} V(\tilde{p}) &= V^0, \\ F(\tilde{p}) &= -(J^0)^{-1} K^0. \end{aligned}$$

Obviously, T should be chosen so large that V^0, J^0, and K^0 are insensitive to T. Thus, given any $\tilde{p}$, the central bank can through this iteration determine the optimal policy. In future periods $t+\tau$ for small $\tau \geq 0$, if there is no new information, the relevant future probability distribution is given by $p_{t+\tau} = \tilde{p}_\tau$, and the corresponding $V(\tilde{p}_\tau)$ and $F(\tilde{p}_\tau)$ are given by

$$\begin{aligned} V(\tilde{p}_\tau) &= V^\tau, \\ F(\tilde{p}_\tau) &= -(J^\tau)^{-1} K^\tau. \end{aligned}$$

However, for larger τ, the corresponding V^τ, J^τ, and K^τ would start being sensitive to T—that is, $T - \tau$ would not be sufficiently large—and the iteration should be redone. Furthermore, any additional information or judgment may lead to the relevant probability distribution in period $t+\tau$ to deviate from $\tilde{p}_\tau$, in which case the iteration (E.1)-(E.6) also needs to be redone.[9]

[9] A related paper is do Val and Basar [5], who consider the problem of "receding horizon control." They introduce a terminal payoff, and at each date t they solve a finite-horizon optimization problem looking ahead T periods given the current probability distribution. The action taken at the current date is then the first optimal choice in the solution of the finite horizon problem. Then the distribution is updated and the problem repeats.

References

[1] Anderson, Evan W., Lars Peter Hansen, Ellen R. McGrattan, and Thomas J. Sargent (1995), "Mechanics of Forming and Estimating Dynamic Linear Economies," working paper.

[2] Brash, Donald T. (2000), "Making Monetary Policy: A Look Behind the Curtains," speech in Christchurch, January 26, 2000, Reserve Bank of New Zealand, www.rbnz.govt.nz.

[3] Costa, O.L.V., and M.D. Fragoso (1995), "Discrete-Time LQ-Optimal Control Problems for Infinite Markov Jump Parameter Systems," *IEEE Transactions on Automatic Control* 40, 2076–2088.

[4] Costa, O.L.V., M.D. Fragoso, and R.P. Marques (2005), *Discrete-Time Markov Jump Linear Systems*, Springer, London.

[5] do Val, João B.R., and Tamer Başar (1999), "Receding Horizon Control of Jump Linear Systems and a Macroeconomic Policy Problem," *Journal of Economic Dynamics and Control* 23, 1099–1131.

[6] do Val, J.B.R, J.C. Geromel, and O.L.V. Costa (1998), "Uncoupled Riccati Iterations for the Linear Quadratic Control Problem of Discret-Time Markov Jump Linear Systems," *IEEE Transactions on Automatic Control* 43, 1727–1733.

[7] Hamilton, James D. (1994), *Time Series Analysis*, Princeton University Press.

[8] Kim, Chang-Jin and Charles R. Nelson (1999), *State-Space Models with Regime Switching*, MIT Press, Cambridge, MA.

[9] Lindé, Jesper (2002), "Estimating New-Keynesian Phillips Curves: A Full Information Maximum Likelihood Approach," Sveriges Riksbank Working Paper Series No. 129.

[10] Ljungqvist, Lars, and Thomas J. Sargent (2005), *Recursive Macroeconomic Theory*, 2nd edition, MIT Press, Cambridge, MA.

[11] Marcet, Albert, and Ramon Marimon (1998), "Recursive Contracts," working paper.

[12] Onatski, Alexei, and Noah Williams (2003), "Modeling Model Uncertainty," *Journal of the European Economic Assocation* 1, 1087–1022.

[13] Rudebusch, Glenn D., and Lars E.O. Svensson (1999), "Policy Rules for Inflation Targeting," in John B. Taylor (ed.), *Monetary Policy Rules*, University of Chicago Press.

[14] Sims, Christopher A. (2002), "The Role of Models and Probabilites in the Monetary Policy Process," *Brookings Papers on Economic Activity* 2:2002, 1–62.

[15] Svensson, Lars E.O. (2001), *Independent Review of the Operation of Monetary Policy in New Zealand: Report to the Minister of Finance*, available at www.princeton.edu/∼svensson.

[16] Svensson, Lars E.O. (2005a), "Monetary Policy with Judgment: Forecast Targeting," *International Journal of Central Banking*, forthcoming, www.princeton.edu/∼svensson.

[17] Svensson, Lars E.O. (2005b), "Optimization under Commitment and Discretion, the Recursive Saddlepoint Method, and Targeting Rules and Instrument Rules: Lecture Notes," www.princeton.edu/∼svensson.

[18] Svensson, Lars E.O., and Michael Woodford (2005), "Implementing Optimal Policy through Inflation-Forecast Targeting," in Bernanke, Ben S., and Michael Woodford, eds., *Inflation Targeting*, University of Chicago Press, www.princeton.edu/∼svensson.

[19] Woodford, Michael (2003), *Interest and Prices*: *Foundations of a Theory of Monetary Policy*, Princeton University Press.