

# Interbank Market Liquidity and Central Bank Intervention\*

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## Abstract

We develop a simple model of the interbank market where banks trade a long term, safe asset. We show that when there is a lack of opportunities for banks to hedge aggregate and idiosyncratic liquidity shocks, the interbank market is characterized by excessive price volatility. In such a situation, a central bank can implement the constrained efficient allocation by using open market operations to fix the short term interest rate. The model shows also that market freezes, where banks stop trading with each other, can be a feature of the constrained efficient allocation if there is sufficient uncertainty about aggregate liquidity demand compared to idiosyncratic liquidity demand.

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# Interbank Market Liquidity and Central Bank Intervention

## Abstract

We develop a simple model of the interbank market where banks trade a long term, safe asset. We show that when there is a lack of opportunities for banks to hedge aggregate and idiosyncratic liquidity shocks, the interbank market is characterized by excessive price volatility. In such a situation, a central bank can implement the constrained efficient allocation by using open market operations to fix the short term interest rate. It is also shown that market freezes, where banks stop trading with each other, can be a feature of the constrained efficient allocation if there is sufficient uncertainty about aggregate liquidity demand compared to idiosyncratic liquidity demand.

# 1 Introduction

Interbank markets are among the most important in the financial system. They allow liquidity to be readily transferred from banks with a surplus to banks with a deficit. They are the focus of central banks' implementation of monetary policy and have a significant effect on the whole economy. Under normal circumstances the interbank markets, especially the short term ones, work rather well. Given the high number of participants and the safe type of assets often used in the transactions, the interbank markets are quite competitive and issues of adverse selection and moral hazard associated with problems of asymmetric information do not seem to play an important role. On occasion, however, such as in the crisis that started in the summer of 2007, even the short-term interbank markets stop functioning well thus inducing central banks to intervene massively in order to try to restore normal conditions.

Despite their apparent importance, interbank markets have received relatively little attention in the academic literature. In particular, there is so far no widely accepted theoretical analysis of how they operate and of what type of imperfection may disrupt their functioning. The purpose of this paper is to develop a simple theoretical framework for analyzing interbank markets and how the central bank should intervene. Our starting point is that banks use the interbank market to hedge themselves against liquidity shocks. However, when hedging opportunities are limited so that markets are incomplete, banks cannot insure themselves completely and the interbank market may exhibit excessive price volatility. By using open market operations appropriately to fix interest rates, the central bank can prevent the price volatility and implement the constrained efficient solution. Thus, the central bank effectively completes the market, and open market operations are sufficient to deal with systemic liquidity crises as argued by Goodfriend and King (1988).

Our analysis is based on a standard banking model developed in Allen and Gale (2004a,b) and Allen and Carletti (2006, 2008). There are two periods in the usual way. Banks can hold one-period liquid assets or two-period long term assets with a higher return. All assets

are risk free in the sense that their promised payoffs are always paid. Banks face uncertain liquidity demands from their customers at the end of the first period. We distinguish between two types of uncertainty concerning banks' liquidity needs. The first is *idiosyncratic uncertainty* that arises from the fact that for any given level of aggregate demand for liquidity there is uncertainty about which banks will face the demand. The basic role of interbank markets is to allow reallocations of liquidity from banks with an excess to banks with a deficit. The second is the *aggregate uncertainty* that is due to the fact that the overall level of the demand for liquidity that banks face is stochastic.

We start with the analysis of the optimal portfolio of assets and payments that a planner who can transfer liquidity costlessly would implement. We assume that the planner is constrained in the same way as banks to offer deposit contracts where the payment at the end of the first period cannot be made contingent on the aggregate demand for liquidity in the banking system or the bank's individual liquidity demand. The resulting optimal allocation is termed the *constrained efficient allocation* because of this constraint to use deposit contracts.

We next consider the operation of an interbank market where banks can buy and sell the long term asset at the end of the first period. Since all assets are risk free in our model, there is no difference between selling the long asset and using it as collateral in a repurchase agreement. For ease of exposition, we consider outright sales of assets. The interbank market allows reallocations of liquidity between banks that depend on the realizations of the idiosyncratic and aggregate liquidity shocks. We focus on situations where the uncertainty concerning liquidity demand is not sufficient to cause banks to fail. In other words, banks find it optimal to keep enough liquidity to insure themselves against the high aggregate liquidity shock. The aggregate uncertainty about liquidity demand leads to volatile equilibrium prices for the long asset at the end of the first period, or equivalently interest rates. The intuition hinges on the simple fact that prices in the interbank market have to adjust to satisfy the market clearing condition and to provide banks with the appropriate incentives to keep the

necessary liquidity initially. When the aggregate liquidity demand turns out to be low (that is, in the good state), there is an excess supply of aggregate liquidity at the end of the first period. The price of the long term asset is bid up to the level where the return during the second period is the same for both assets so that banks will be willing to hold both of them. The high price in the good state implies that prices have to fall in the bad state, that is when the high aggregate liquidity shock is realized, in order for banks to be willing to hold both the short and the long term assets initially. If this was not the case, the long asset would dominate the short asset and banks would not hold any liquidity to start with. Given that consumers are risk averse, this price volatility is inefficient because it leads to consumption volatility across states thus preventing the implementation of the constrained efficient allocation.

The main result of the paper is to show that the introduction of a central bank that engages in open market operations to fix the price of the long asset at the end of the first period (or equivalently fix the short term interest rate) removes the inefficiency associated with a lack of hedging opportunities. This intervention allows the banks to implement the constrained efficient allocation. To see how this occurs it is helpful to consider two special cases. The first is where there is just idiosyncratic liquidity risk and no aggregate risk. Provided the central bank engages in the right open market operations and fixes the price in the interbank market at the end of the first period at the appropriate level, banks with a high liquidity demand will be able to sell their holdings of the long term asset to raise liquidity. The banks with low liquidity demand at the end of the first period are happy to buy the long asset and provide liquidity to the market because they need payoffs at the end of the second period to meet their needs then.

The second special case is where there is no idiosyncratic uncertainty but there is aggregate uncertainty about liquidity demand. Here the central bank must fix the price by engaging in open market operations. In particular, it needs to remove excess liquidity from the banks by selling the long asset when aggregate liquidity demand is low. It can do this

by selling government securities that replicate the long asset that are funded through lump sum taxes on late consumers at the final date. The optimal intervention by the central bank when there is both idiosyncratic and aggregate uncertainty combines the two policies in the special cases. The central bank must fix the price at the appropriate level that allows banks to reallocate liquidity from those with low idiosyncratic shocks to those with high ones. At the same time it must use open market operations to control the aggregate liquidity in the market to fix the price. We show that achieving both objectives simultaneously is possible and the constrained efficient allocation can be implemented. This result is in line with the argument of Goodfriend and King (1988) that open market operations are sufficient to address pure liquidity risk on the interbank market.

One of the implications of our model is that even when the constrained efficient allocation is being implemented by the policies of the central bank, an increase in aggregate uncertainty can cause banks to stop using the interbank markets to trade with each other. The banks hoard liquidity because they may need it to meet high aggregate demand. When aggregate demand is low, however, they have enough liquidity to deal with variations in idiosyncratic demand and as a result the market “freezes”. At least in the context of the model considered here, where the market freezing does not have consequences on the banks’ ability to remain active, there is no need for central banks to intervene to try and unfreeze the markets since the freeze is consistent with constrained efficiency.

The basic problem in our model that leads to a need for central bank intervention is that financial markets are incomplete. In particular, banks are unable to hedge the idiosyncratic and aggregate liquidity shocks that they face. In the remaining part of the paper we consider how complete markets would operate and allow these risks to be hedged. There are many forms that such complete markets could take. We consider how markets for Arrow securities where all trades are made at the initial date allow the constrained efficient allocation to be implemented. We also show how a sequence of markets at the initial date followed by a market for second period consumption at the end of the first period can also implement the

constrained efficient allocation. Both of these cases involve a large number of securities being issued and traded. In practice, the costs of issuance and of the infrastructure for trading securities to implement such a system are likely to be prohibitive.

Our paper is not the only one to consider inefficiencies in the interbank market. In a recent contribution, Acharya, Gromb and Yorulmazer (2008) model the interbank markets as being characterized by moral hazard, asymmetric information, and monopoly power in times of crisis. In their model, a bank with surplus liquidity is able to bargain with a bank that needs liquidity to keep funding projects. This bargaining allows the surplus bank to extract surplus from the deficit bank and this results in an inefficient allocation of resources. The role of the central bank in their model is to provide an outside option to the deficit bank for acquiring the needed liquidity. Even if the central bank does not actually provide the liquidity the inefficient bargaining can be avoided. The authors provide a number of historical examples where some banks had monopoly power over others in times of crisis. An important issue is how relevant their analysis is for modern interbank markets. Still focusing on problems of asymmetric information, Heider, Hoerova and Holthausen (2008) find that when a bank's credit risk cannot be directly observed safer borrowers drop out of the interbank market and lenders hoard liquidity despite the high prevailing interest rate when the counterparty risk in the market rises sufficiently. Differently from both of these papers, we abstract from issues of asymmetric information and analyze situations where the pure inability of banks to hedge themselves against liquidity shocks can lead to inefficiencies in the equilibrium prices.

Several other papers have studied the functioning of interbank markets. Bhattacharya and Gale (1987) show that banks can optimally cope with idiosyncratic liquidity shocks by borrowing and lending liquidity; but tend to under-invest in liquidity reserves when moral hazard and adverse selection problems are present. Allen and Gale (2000) show that interbank markets provide optimal liquidity insurance when banks are subject to idiosyncratic shocks, but may lead to contagion when aggregate shocks are present and connections among

banks are limited. In a similar spirit, Freixas et al. (2000) analyze the risk of contagious runs through the payment system when banks are located in different regions and face both liquidity and solvency shocks. Other reasons for the poor functioning of the interbank market relate to asymmetric information (Flannery, 1996; and Freixas and Jorge, 2007) and banks' free riding on central bank liquidity (Repullo, 2005).

There are also a number of related papers where assets are liquidated in markets that do not work properly and some government intervention may be needed. The optimal form of intervention depends on the reason why the liquidation markets do not allocate liquidity efficiently. Holmstrom and Tirole (1998) and Diamond and Rajan (2005) analyze the optimal liquidity provision by a central bank or a government authority when interbank markets are subject to aggregate liquidity shocks and contagious failures generated by the illiquidity of bank assets. Gorton and Huang (2004) show that government may optimally supply liquidity by issuing government bonds when banks need to sell distressed assets in an illiquid market. Gorton and Huang (2006) explain the lender of last resort function of central banks with the need of monitoring banks and providing them with liquidity in times of crises in order to prevent inefficient panics. In a context where banks may herd and generate banking crises by forcing a reduction in bank asset prices, Acharya and Yorulmazer (2008) show that it may be optimal for the regulator to bail out some failed banks. Such an ex post policy, however, is dominated by an ex ante liquidity assistance policy to the surviving banks in the purchase of failed banks.

The main difference between all of these analyses and ours is that they focus on very different types of market failure based on asymmetric information, moral hazard, and monopoly power. In practice interbank markets tend to be highly competitive, efficient markets. Transactions often involve transactions in safe government securities or are in the form of repurchase agreements that are collateralized with safe government securities so that the importance of asymmetric information and moral hazard are limited. In contrast, our analysis is based on incomplete markets that result in limited hedging opportunities for banks. The

model is consistent with competitive and liquid markets where participants have symmetric information. Moreover, most of these papers do not consider the type of central bank intervention that is actually observed in interbank markets where central banks use open market operations to fix short term interest rates. This type of intervention is the focus of our analysis.

The paper proceeds as follows. Section 2 describes the model. The constrained efficient allocation is considered in Section 3. We then consider the operation of an interbank market for the long asset in Section 4. The role of the central bank is analyzed in Section 5. Section 7 considers how complete markets would implement the constrained efficient allocation. Finally, Section 8 concludes.

## 2 The model

The model is based on Allen and Gale (2004a,b) and Allen and Carletti (2006, 2008). There are three dates  $t = 0, 1, 2$  and a single, all-purpose good that can be used for consumption or investment at each date. The banking sector consists of a large number of competitive institutions.

There are two securities, one *short* and one *long*. Both are risk free. The short security is represented by a storage technology: one unit at date  $t$  produces one unit at date  $t + 1$ . The long security is a simple constant-returns-to-scale investment technology that takes two periods to mature: one unit invested in the long security at date 0 produces  $R > 1$  units of the good at date 2 so it is more productive than the short security.

We assume there is a market for liquidating the long asset at date 1. Each unit can be sold for  $P$ . Participation in this market is limited: financial institutions such as banks can buy and sell in the asset market at date 1 but individual consumers cannot.

Banks raise funds from depositors, who have an endowment of one unit of the good at date 0 and none at dates 1 and 2. Depositors are uncertain about their preferences:

with probability  $\lambda$  they are *early consumers*, who only value the good at date 1, and with probability  $1 - \lambda$  they are *late consumers*, who only value the good at date 2. There are two types of uncertainty that determine  $\lambda$  :

$$\lambda_{\theta i} = \alpha_i + \varepsilon\theta$$

where  $\alpha_i, i = H, L$  is an *idiosyncratic* bank-specific shock and  $\theta = 0, 1$  is an *aggregate shock*. Except where otherwise stated we assume  $\varepsilon > 0$ . For simplicity, we assume that the random variables  $\alpha_i$  and  $\theta$  have two-point supports. That is:

$$\alpha_H = \bar{\alpha} + \eta \quad \text{w. pr. } 0.5,$$

$$\alpha_L = \bar{\alpha} - \eta \quad \text{w. pr. } 0.5,$$

where  $0 < \alpha_L \leq \alpha_H < 1$ ; and

$$\theta = \begin{cases} 0 & \text{w. pr. } \pi, \\ 1 & \text{w. pr. } (1 - \pi), \end{cases}$$

where  $0 < \pi < 1$ . Because there are only two values of  $\theta$ , the price at which the long asset can be sold at date 1 takes at most two values,  $P_\theta$ , where  $\theta = 0, 1$ .

Uncertainty about time preferences generates a preference for liquidity and a role for the intermediary as a provider of liquidity insurance. The utility of consumption is represented by a utility function  $u(c)$  with the usual properties. Expected utility at date 0 is given by

$$EU = E [\lambda u(d) + (1 - \lambda)u(c)],$$

where  $c_t$  denotes consumption at date  $t = 1, 2$ .

Banks compete by offering deposit contracts to consumers in exchange for their endowments and consumers respond by choosing the most attractive of the contracts offered. Free entry ensures that banks earn zero profits in equilibrium. The deposit contracts offered in

equilibrium must maximize consumers' welfare subject to the zero-profit constraint. Otherwise, a bank could enter and make a positive profit by offering a more attractive contract.

There is no loss of generality in assuming that consumers deposit their entire endowment in a bank at date 0 since the bank can do anything the consumers can do. The bank invests  $y$  units per capita in the short asset and  $1 - y$  units per capita in the long asset and offers each consumer a deposit contract, which allows the consumer to withdraw either  $d$  units at date 1 or the residue of the bank's assets at date 2 divided equally among the remaining depositors.

A consumer's type is private information. An early consumer cannot misrepresent his type because he needs to consume at date 1; but a late consumer can claim to be an early consumer, withdraw  $d$  at date 1, store it until date 2 and then consume it. The deposit contract is *incentive compatible* if and only if the residual payment to late consumers at date 2 is at least  $d$ . Since the late consumers are residual claimants at date 2, it is possible to give them at least  $d$  units of consumption if and only if

$$\lambda d + (1 - \lambda)d \frac{P_\theta}{R} \leq y + P_\theta(1 - y). \quad (1)$$

The left hand side is a lower bound for the present value of consumption at date 1 when early consumers are given  $d$  and late consumers are given at least  $d$ . The first term is the consumption given to the early consumers. The second term is the present value of the  $(1 - \lambda)d$  given to the late consumers. The price of the long asset at date 1 is  $P_\theta$  and this long asset pays off  $R$  at date 2 so the date 1 present value of 1 unit of consumption at date 2 is  $P_\theta/R$ . The right hand side is the value of the bank's portfolio. The bank has  $y$  in the short asset and  $(1 - y)$  of the long asset worth  $P_\theta$  per unit. Thus, condition (1) is necessary and sufficient for the deposit contract  $d$  to satisfy incentive compatibility and the budget constraint simultaneously. If (1) was not satisfied the late consumers would receive less than the early consumers if they left their funds in the bank so they would find it optimal to

withdraw and there would be a run. The inequality in (1) is referred to as the *incentive constraint* for short. We restrict our analysis to the set of parameters where this constraint is satisfied for the optimal contract. We also assume that bank runs do not occur when the constraint is satisfied. In other words, late consumers will withdraw at date 2 as long as the bank can satisfy the incentive constraint.

All uncertainty is resolved at the beginning of date 1. In particular, depositors learn whether they are early or late consumers and the values of  $\alpha$  and  $\theta$  are determined. While each depositor's individual realization of liquidity demand is observed only by them,  $\alpha$  and  $\theta$  are publicly observed.

### 3 The constrained efficient allocation

The planner invests in a portfolio of the short and long asset. The proceeds are distributed directly to early and late consumers. The planner does not need to worry about idiosyncratic liquidity risk since the  $H$  group with  $\alpha_H$  early consumers will be balanced by the  $L$  group with  $\alpha_L$  early consumers. It is possible to just plan for  $\bar{\alpha}$  early consumers in total.

The planner provides early consumers with consumption  $d$  and late consumers receive  $c_{20}$  when  $\theta = 0$  and  $c_{21}$  when  $\theta = 1$ . Using the notation  $\lambda_0 = \bar{\alpha}$  and  $\lambda_1 = \bar{\alpha} + \varepsilon$  the planner's problem can be written

$$\begin{aligned}
& \max_{y, d} && \pi [\lambda_0 u(d) + (1 - \lambda_0)u(c_{20})] + (1 - \pi) [\lambda_1 u(d) + (1 - \lambda_1)u(c_{21})] \\
& && \lambda_0 d \leq y \\
& && (1 - \lambda_0)c_{20} = y - \lambda_0 d + (1 - y)R \\
& \text{s.t.} && \lambda_1 d \leq y \\
& && (1 - \lambda_1)c_{21} = y - \lambda_1 d + (1 - y)R \\
& && 0 \leq d, 0 \leq y \leq 1.
\end{aligned} \tag{2}$$

The first two constraints represent the physical constraints on consumption at the two

dates in state  $\theta = 0$ . At date 1 it is not possible to consume more output than exists. At date 2 the  $(1 - \lambda_0)$  late consumers consume  $c_{20}$ . The total amount available for them is whatever is not consumed at date 1,  $y - \lambda_0 d$ , together with what is produced at date 2,  $(1 - y)R$ . Similarly for the next two constraints for state  $\theta = 1$ . Finally, we have the usual constraints on  $d$  and  $y$ .

We denote the optimal solution to this problem  $y^*$  and  $d^*$ . Note that it cannot be the case at the optimum that  $y^* > \lambda_1 d^*$ . If this inequality held, it would be possible to increase expected utility by holding  $d$  constant and reducing  $y$  since  $R > 1$ . Hence at the optimum

$$y^* = \lambda_1 d^* > \lambda_0 d^*. \quad (3)$$

Thus the planner's problem is to choose  $d$  to

$$\max \pi \left[ \lambda_0 u(d) + (1 - \lambda_0) u\left(\frac{\varepsilon d + (1 - \lambda_1 d)R}{1 - \lambda_0}\right) \right] + (1 - \pi) \left[ \lambda_1 u(d) + (1 - \lambda_1) u\left(\frac{(1 - \lambda_1 d)R}{1 - \lambda_1}\right) \right].$$

This gives the first order condition that determines  $d^*$

$$\pi \left[ \lambda_0 u'(d^*) + u'\left(\frac{\varepsilon d^* + (1 - \lambda_1 d^*)R}{1 - \lambda_0}\right)(\varepsilon - \lambda_1 R) \right] + (1 - \pi) \left[ \lambda_1 u'(d^*) + u'\left(\frac{(1 - \lambda_1 d^*)R}{1 - \lambda_1}\right)(-\lambda_1 R) \right] = 0. \quad (4)$$

Differentiating a second time with respect to  $d$  it can be seen that

$$\pi \left[ \lambda_0 u''(d) + u''(c_{20}) \frac{(\varepsilon - \lambda_1 R)^2}{1 - \lambda_0} \right] + (1 - \pi) \left[ \lambda_1 u''(d) + u''(c_{21}) \frac{(\lambda_1 R)^2}{1 - \lambda_1} \right] < 0$$

since  $u'' < 0$ . Thus the constrained efficient allocation is unique.

In the special case of no aggregate risk where  $\varepsilon = 0$  the first order condition simplifies to

$$u'(d^*) = u'\left(\frac{(1 - \lambda d^*)R}{1 - \lambda}\right)R. \quad (5)$$

When  $u = \log_e(c_t)$  we have the further simplification

$$y^* = \lambda, d^* = 1, c_2^* = R.$$

With constant relative risk aversion utility it can be shown that  $d^* < 1$  if relative risk aversion is less than one and  $d^* > 1$  if it is greater than one (see, e.g., pp. 68-69 of Allen and Gale (2007)).

We turn next to consider the allocation when there is an interbank market at date 1 that allows banks to buy and sell the long asset.

## 4 Interbank markets

Suppose there is an interbank market at date 1 for trading the long asset at price  $P_\theta$ . Banks can buy the long and short assets at date 0 for a price of 1 and at date 1 it is also possible to buy the short term asset at a price of 1. This set of markets is *incomplete* in that it is not possible to completely hedge the risk of aggregate and idiosyncratic liquidity shocks. It is shown that this incompleteness leads to price volatility.

Once the banks have received the funds of depositors at date 0 they can use them to obtain the long and the short assets. In addition to choosing their portfolio of  $y$  in the safe asset and  $1 - y$  in the long asset at date 0, they must also set the amount  $d$  that depositors can withdraw at date 1. Once they know the level of aggregate liquidity demand and their own idiosyncratic liquidity shock at date 1, they can use the interbank market to buy or sell the long asset.

The consumption of a bank's depositors at date 2 depends on the aggregate state since this determines  $P_\theta$ . It also depends on the idiosyncratic shock that strikes the bank since this determines the proportions  $\lambda$  of early and  $1 - \lambda$  of late consumers. In particular, for  $y$

and  $d$  such that the incentive constraint (1) is satisfied so bankruptcy is avoided

$$c_{2\theta i} = \frac{\left[1 - y + \frac{y - \lambda_{\theta i} d}{P_{\theta}}\right] R}{1 - \lambda_{\theta i}}, \quad (6)$$

for  $\theta = 0, 1$  and  $i = H, L$ . The term in square brackets represents the amount of long asset held by the bank at date 2. The  $(1 - y)$  term is the initial holding of the long asset purchased at date 0. If  $y - \lambda_{\theta i} d > 0$  then excess liquidity at date 1 can be used to purchase the long asset. The amount of the long asset that can be purchased is  $(y - \lambda_{\theta i} d)/P_{\theta}$ . If  $y - \lambda_{\theta i} d < 0$  then it is necessary to sell the long asset held by the bank in the market at date 1 to fund the shortfall of liquidity. In this case  $(y - \lambda_{\theta i} d)/P_{\theta}$  represents the amount that must be sold. Each unit of the long asset pays off  $R$  and the total payoff must be split between the  $(1 - \lambda_{\theta i})$  late consumers.

For  $y$  and  $d$  such that the incentive constraint (1) is not satisfied so there is a run on the bank and it has to liquidate all of its assets at date 1

$$c_{2\theta i} = y + (1 - y)P_{\theta}, \quad (7)$$

for  $\theta = 0, 1$  and  $i = H, L$ . The first term  $y$  is the payoff of the short asset and the second term  $(1 - y)P_{\theta}$  is what is obtained from liquidating the long asset in the interbank market at date 1. As explained above, we focus on the case where the incentive constraint is satisfied at the bank's optimal choice. We will therefore take  $c_{2\theta i}$  to be given by (6) below.

The problem each bank solves at date 0 is to choose  $y$  and  $d$  to

$$\begin{aligned} \max \quad & 0.5\{\pi[\lambda_{0H}u(d) + (1 - \lambda_{0H})u(c_{20H}) + \lambda_{0L}u(d) + (1 - \lambda_{0L})u(c_{20L})] \\ & + (1 - \pi)[\lambda_{1H}u(d) + (1 - \lambda_{1H})u(c_{21H}) + \lambda_{1L}u(d) + (1 - \lambda_{1L})u(c_{21L})]\} \\ \text{s.t.} \quad & 0 \leq d, 0 \leq y \leq 1, \end{aligned} \quad (8)$$

taking prices  $P_0$  and  $P_1$  as given. The first order conditions for this with respect to the choice of  $y$  and  $d$  are:

$$\pi \left( \frac{1}{P_0} - 1 \right) [u'(c_{20H}) + u'(c_{20L})] + (1 - \pi) \left( \frac{1}{P_1} - 1 \right) [u'(c_{21H}) + u'(c_{21L})] = 0 \quad (9)$$

$$\begin{aligned} & [\bar{\alpha} + (1 - \pi)\varepsilon]u'(d) - 0.5R \left[ \frac{\pi}{P_0} (\alpha_H u'(c_{20H}) + \alpha_L u'(c_{20L})) \right. \\ & \left. + \frac{1 - \pi}{P_1} ((\alpha_H + \varepsilon)u'(c_{21H}) + (\alpha_L + \varepsilon)u'(c_{21L})) \right] = 0. \end{aligned} \quad (10)$$

Now since the aggregate measure of banks is 1 the aggregate amount of liquidity is  $y$ . There are two aggregate states of demand for liquidity,  $\theta = 0$  where  $\lambda_0 = \bar{\alpha}$  and  $\theta = 1$  where  $\lambda_1 = \bar{\alpha} + \varepsilon$ . Within each of these states, half of the banks have high idiosyncratic demand,  $\alpha_H$ , for liquidity. In this case they can liquidate part of their holdings of the long asset in the interbank market to meet the high demand for liquidity from their customers. The other half of the banks have low liquidity demand,  $\alpha_L$ . They are willing to use their excess liquidity to buy the long asset in the interbank market. Since, we are assuming bankruptcy is not optimal, we know that the aggregate amount of liquidity  $y$  must be sufficient to cover demand in state  $\theta = 1$  so we have

$$y \geq (\bar{\alpha} + \varepsilon)d.$$

Since  $\varepsilon > 0$  this implies that

$$y > \bar{\alpha}d.$$

As a result there is excess liquidity at date 1 in state  $\theta = 0$ . In order for the interbank market to clear it is necessary that

$$P_0 = R. \quad (11)$$

In this case banks are willing to hold both the long asset and the excess liquidity between dates 1 and 2. If  $P_0 < R$  they will be willing to hold only the long asset while if  $P_0 > R$  they will be willing to hold only the short asset. Hence  $P_0$  must be given by (11).

Notice that if  $y > (\bar{\alpha} + \varepsilon)d$  a similar argument would hold for state  $\theta = 1$  and we would have  $P_1 = R$ . But this cannot be an equilibrium given  $P_0 = R$  because then the long asset would dominate the short asset between dates 0 and 1 and there would be no investment in the short asset at all. Hence equilibrium requires

$$y = (\bar{\alpha} + \varepsilon)d = \lambda_1 d. \quad (12)$$

It then follows that  $P_1$  must be such that banks are willing to hold both the long and short asset between dates 0 and 1. To find the equilibrium value of  $P_1$  we substitute for  $P_0$  and  $y$  using (11) and (12) and solve the first order conditions (9) and (10) for  $P_1$  and  $d$ .

An important issue concerns the circumstances under which the interbank market “freezes” or in other words when the banks will stop trading with each other. The essential purpose of the interbank market is to allow banks with high liquidity needs to sell the long asset and obtain liquidity from banks with low liquidity needs. If the amount of liquidity the banks hold to deal with aggregate uncertainty is large enough then in state  $\theta = 0$  when aggregate liquidity demand is low, they may not need to go to the interbank market to raise liquidity since they hold so much internally anyway. In particular, they will not need to enter the market in state  $\theta = 0$  when they are an  $H$  bank if

$$\lambda_1 d > \lambda_{0H} d.$$

Using  $\lambda_1 = \bar{\alpha} + \varepsilon$  and  $\lambda_{0H} = \bar{\alpha} + \eta$  it can be seen that this simplifies to

$$\varepsilon > \eta.$$

Thus the market will freeze if aggregate uncertainty is large enough relative to idiosyncratic uncertainty.

## 5 Central bank intervention

In this section we introduce a central bank that can engage in open market operations. In practice central banks hold large portfolios of securities that they use to intervene in the markets. They buy or sell securities to affect the amount of liquidity held by banks. In recent years the focus of most central banks has been to use open market operations to target the interest rate in the overnight interbank market. In order to explain how the central bank can implement the constrained efficient allocation, we proceed in three steps. The first is to show how this can be done when there is only idiosyncratic risk. The second is to show how open market operations can be used when there is just aggregate risk. Finally, we consider the two types of risk together.

### 5.1 Idiosyncratic liquidity risk alone: $\eta > 0, \varepsilon = 0$

We start with the simplest case where there is only idiosyncratic risk in liquidity demand, and no aggregate risk so  $\eta > 0, \varepsilon = 0$ . It was shown in Section 3 that in the special case of log utility where  $u(c) = \log_e c$  the constrained efficient allocation is

$$y = \bar{\alpha}; d = 1; c_2 = R.$$

Allen and Gale (2004b) show that in this case if there is a non-stochastic price, then there is a unique equilibrium with  $P = 1$  that corresponds to the constrained efficient allocation. To see that this is an equilibrium note that if  $P > 1$ , then the long asset dominates, while if  $P < 1$  the short asset dominates. Only if  $P = 1$  are banks willing to hold both assets between dates 0 and 1. Allen and Gale also show that there are many other sunspot equilibria with random prices. Since there is idiosyncratic risk the banks trade at these prices and this leads to an allocation that is worse in an ex ante sense than the constrained efficient allocation. Given this multiplicity of equilibria, by setting  $P = 1$  the central bank ensures that the constrained efficient allocation is implemented. In this equilibrium banks that have a high

liquidity shock  $H$  sell the long asset and banks with low liquidity shock  $L$  buy it. Both types of banks can afford to implement the optimal allocation at date 1. One unit of the short asset enables them to provide early consumers with  $d = 1$  while one unit of the long asset, which costs the same, allows them to provide the late consumers with  $c_2 = R$ . Any composition of their depositors between early and late can therefore be accommodated.

This case is, of course, very special because of the log utility function. The fact that  $d = 1$  and  $c_2 = R$  leads to the special property that banks are indifferent between having early and late consumers. We next show that by holding an appropriate portfolio of securities and engaging in open market operations and fixing the price of the long asset at  $P = 1$ , the central bank can ensure that this property holds more generally.

Let  $y^*$ ,  $d^*$ , and  $c_2^*$  denote the constrained efficient allocation as before. Since there is no aggregate uncertainty we know that it is efficient to use the short asset to provide early consumption and the long asset to provide late consumption so

$$\begin{aligned} y^* &= \bar{\alpha}d^* = \lambda d^* \\ c_2^* &= \frac{(1 - y^*)R}{1 - \lambda}. \end{aligned}$$

Our approach is to show that the banks can provide their depositors with this allocation provided the central bank adopts the optimal policy. We also show that it is individually optimal for each bank to choose it.

Let  $X_0$  denote the lump sum tax that is imposed by the government at date 0 to fund the portfolio for open market operations of the central bank. The central bank uses these funds to buy the short term asset at date 0. Depositors then have  $1 - X_0$  remaining that they put in the banks. Suppose the banks hold  $y^* - X_0$  in the short asset and  $1 - y^*$  in the long asset between dates 0 and 1.

At date 1, half the banks have  $\lambda_H = \bar{\alpha} + \eta$  early consumers while the other half have  $\lambda_L = \bar{\alpha} - \eta$ . Banks of type  $i, i = H, L$  require total liquidity of  $\lambda_i d^*$ . They have liquidity

$y^* - X_0$  so their net need is  $y^* - X_0 - \lambda_i d^*$ . If this is positive they use it to buy the long term asset. If it is negative they sell the long term asset to raise the needed liquidity. The central bank sets  $P = 1$  and supplies its holding of the short asset  $X_0$  to the market and receives  $X_0$  of the long asset. The interbank market clears since

$$0.5(y^* - X_0 - \lambda_H d^*) + 0.5(y^* - X_0 - \lambda_L d^*) + X_0 = y^* - \bar{\alpha} d^* = 0.$$

A bank of type  $i$  now has  $1 - y^* + y^* - X_0 - \lambda_i d^* = 1 - X_0 - \lambda_i d^*$  in the long asset. At date 2 these holdings allow the banks to provide a payout to their late consumers of

$$\beta_{2i} = \frac{(1 - X_0 - \lambda_i d^*)R}{1 - \lambda_i}.$$

At date 2, the central bank has  $X_0 R$ . These funds are remitted to the government and the government then distributes them as a lump sum grant to the  $1 - \lambda$  late consumers of

$$\gamma_2 = \frac{X_0 R}{1 - \lambda}.$$

In order to implement the constrained efficient allocation, it is necessary that the sum of these payouts is equal to  $c_2^*$ . Thus we need

$$\beta_{2i} + \gamma_2 = \frac{(1 - X_0 - \lambda_i d^*)R}{1 - \lambda_i} + \frac{X_0 R}{1 - \lambda} = c_2^* = \frac{(1 - y^*)R}{1 - \lambda}.$$

It can easily be checked that

$$X_0 = 1 - d^*$$

solves this equation for any value of  $\lambda_i$ .

We next need to show that if the central bank implements the optimal policy with  $P = 1$ ,  $X_0 = 1 - d^*$ , and  $\gamma_2 = (1 - d^*)R/(1 - \lambda)$  that if each bank takes this policy as given it is optimal for them to choose the constrained efficient allocation.

Each bank has deposits of  $1 - X_0$ . It invests in  $y$  of the short asset and  $1 - X_0 - y$  of the long asset at date 0. The bank's problem is

$$\begin{aligned} \text{Max}_{y,d} & 0.5[\lambda_H u(d) + (1 - \lambda_H)u(c_{2H})] + 0.5[\lambda_L u(d) + (1 - \lambda_L)u(c_{2L})] \\ \text{s.t. } c_{2i} &= \frac{[1 - X_0 - y + \frac{y - \lambda_i d}{P}]R}{1 - \lambda_i} + \gamma_2 \text{ for } i = H, L. \end{aligned}$$

The objective function is the standard one. The expression for  $c_{2i}$  is the sum of the payoff from the bank and the lump sum grant  $\gamma_2$ . The payoff from the bank is the original holding of the long asset  $1 - X_0 - y$  and the amount purchased (or sold) at date 1 in the interbank market  $(y - \lambda_i d)/P$ . (Note that it is optimal not to hold any of the short asset from date 1 to date 2 provided  $P < R$  which is the case here.) This long asset pays off  $R$  at date 2. The total payoff from the long asset is then divided among the  $1 - \lambda_i$  late consumers of the bank.

Now given the central bank sets  $P = 1$  we have

$$c_{2i} = \frac{[1 - X_0 - \lambda_i d]R}{1 - \lambda_i} + \gamma_2 \text{ for } i = H, L.$$

Thus the choice of  $y$  is irrelevant for each individual bank. We return to the determination of the aggregate level of  $y$  below.

Each bank's problem now simplifies to

$$\begin{aligned} \text{Max}_d & 0.5[\lambda_H u(d) + (1 - \lambda_H)u(\frac{[1 - X_0 - \lambda_H d]R}{1 - \lambda_H} + \gamma_2)] + \\ & 0.5[\lambda_L u(d) + (1 - \lambda_L)u(\frac{[1 - X_0 - \lambda_L d]R}{1 - \lambda_L} + \gamma_2)]. \end{aligned}$$

The first order condition for this problem is

$$0.5[\lambda_H u'(d) + u'(\frac{[1 - X_0 - \lambda_H d]R}{1 - \lambda_H} + \gamma_2)(-\lambda_H R)] +$$

$$0.5[\lambda_L u'(d) + u'(\frac{[1 - X_0 - \lambda_L d]R}{1 - \lambda_L} + \gamma_2)(-\lambda_L R)] = 0.$$

Note that differentiating with respect to  $d$  shows that the second order condition is satisfied. Thus the bank has a unique optimal choice of  $d$ . Substituting for the central bank's optimal policy  $X_0 = 1 - d^*$ , and  $\gamma_2 = (1 - d^*)R/(1 - \lambda)$  gives

$$0.5[\lambda_H u'(d) + u'(\frac{[d^* - \lambda_H d]R}{1 - \lambda_H} + \frac{(1 - d^*)R}{1 - \lambda})(-\lambda_H R)] +$$

$$0.5[\lambda_L u'(d) + u'(\frac{[d^* - \lambda_L d]R}{1 - \lambda_L} + \frac{(1 - d^*)R}{1 - \lambda})(-\lambda_L R)] = 0.$$

If the bank chooses  $d = d^*$  then using  $\lambda = 0.5\lambda_H + 0.5\lambda_L$  the left hand side simplifies to

$$\lambda \left[ u'(d^*) - u'(\frac{(1 - \lambda d^*)R}{1 - \lambda})R \right].$$

It follows from the definition of  $d^*$  from the constrained efficient allocation when there is no aggregate risk (5) that this must be 0. Then  $d = d^*$  is the unique optimal choice for the banks.

It remains to consider how the aggregate value of  $y$  is determined. Market clearing at date 1 requires that  $y = y^* = \lambda d^*$  and this determines the aggregate amount of the short asset in equilibrium.

To sum up, it is not only feasible but is in fact optimal for the banks to implement the constrained efficient allocation.

We have shown that the central bank can use open market operations to implement the constrained efficient allocation. By holding a portfolio of  $1 - d^*$  of the short asset and setting  $P = 1$  the central bank effectively allows the banks to be indifferent to having early or late

consumers. They give early consumers  $d^*$  and late consumers

$$\beta_{2i} = \frac{(1 - X_0 - \lambda_i d^*)R}{1 - \lambda_i} = d^* R.$$

Both cost the same to accommodate in the date 1 market. Thus the size of a bank's idiosyncratic shock becomes irrelevant. We have demonstrated this for two groups  $H$  and  $L$  but it is clear that the same result will hold for an arbitrary distribution of idiosyncratic shocks.

In the discussion so far, we have used the terminology that  $X_0 = 1 - d^*$  is a lump sum tax that finances the central bank's holdings of the short asset from date 0 to date 1. The central bank uses these holdings to purchase the long term asset and holds it between date 1 and date 2. At date 2 the payoffs from this long term asset are paid to the government and they are used to finance a lump sum grant to late consumers. This terminology presumes  $d^* < 1$ . This will be the case if, for example, constant relative risk aversion is sufficiently below 1. However, it is also quite possible that  $d^* > 1$  if, for example, constant relative risk aversion is sufficiently above 1. In this case all the signs are reversed. Instead of a lump sum tax,  $X_0$  represents a lump sum grant of an asset that replicates the short asset. In other words, the asset is "money". The central bank has a liability rather than an asset. At date 1 the central bank sells an asset that replicates the long term asset issued by the government to remove the money from the banking system. At date 2 there is a lump sum tax to make the payment on the government asset replicating the long asset. Thus the theory presented provides a rich model of central bank intervention in the interbank markets. This theory distinguishes between interventions using real assets that are claims on technologies and interventions using liabilities of the government that are money and long term bonds.

We next consider how open market operations can be used to deal with aggregate liquidity risk.

## 5.2 Aggregate liquidity risk alone: $\eta = 0, \varepsilon > 0$

Next consider what happens with no idiosyncratic risk but with positive aggregate risk. In this case the constrained efficient allocation can be implemented by having the central bank engage in open market operations to fix  $P_0 = P_1 = 1$ .

Let  $y^*$ ,  $d^*$ , and  $c_{2\theta}^*$  denote the allocation. We know from Section 3 that

$$y^* = \lambda_1 d^* = (\lambda_0 + \varepsilon) d^*, \quad (13)$$

$$c_{20}^* = \frac{y^* - \lambda_0 d^* + (1 - y^*)R}{1 - \lambda_0} = \frac{\varepsilon d^* + (1 - y^*)R}{1 - \lambda_0}, \quad (14)$$

$$c_{21}^* = \frac{(1 - y^*)R}{1 - \lambda_1}. \quad (15)$$

As usual we show that it is feasible for the banks to implement the constrained efficient allocation. Given that this provides the highest level of expected utility that is possible, if it is feasible it must be optimal for the banks to choose it.

At date 0 the banks hold  $y^*$  of the short asset and  $1 - y^*$  of the long asset.

At date 1 in state  $\theta = 0$  the central bank needs to drain liquidity to ensure  $P_0 = 1$ . The government issues  $X_1 = \varepsilon d^*$  of debt at date 1 that pays  $R$  at date 2. Thus the total owed by the government on its debt at date 2 is  $\varepsilon d^* R$ . The debt is given to the central bank at date 1 to allow it to conduct open market operations to fix the price of the long asset or equivalently the interest rate in the interbank market. In order to do this it sells the government debt, which is equivalent to the long asset, for  $P_0 = 1$ . This removes the excess liquidity from the market and prevents the price of the long asset being bid up to  $P_0 = R$ . The central bank holds the liquidity of  $\varepsilon d^*$  that it acquires until date 2.

After the central bank's open market operations, the banks own  $1 - y^* + \varepsilon d^*$  of the long asset. At date 2 this allows each of them to pay to each of their  $1 - \lambda_0$  late consumers

$$\beta_{20} = \frac{(1 - y^* + \varepsilon d^*)R}{1 - \lambda_0}.$$

The central bank ends up at date 2 with  $X_1 = \varepsilon d^*$  of the short asset. We assume that the proceeds from these assets are returned to the government. The government has resources of  $\varepsilon d^*$  and owes  $\varepsilon d^* R$  on its long term debt so it needs to impose a lump sum tax on each of the  $1 - \lambda_0$  late consumers of

$$\gamma_{20} = \frac{\varepsilon d^*(1 - R)}{1 - \lambda_0}.$$

distributed as a lump sum grant to the late consumers.

Hence late consumers receive

$$\beta_{20} + \gamma_{20} = \frac{\varepsilon d^* + (1 - y^*)R}{1 - \lambda_0} = c_{20}^*$$

as required to implement the constrained efficient allocation.

At date 1 in state  $\theta = 1$  each bank pays  $d^*$  to  $\lambda_1$  early consumers. They have no short asset and  $1 - y^*$  of the long asset after this. The central bank does not need to actively conduct open market operations to ensure  $P_1 = 1$  so it does not intervene. Each bank pays  $(1 - y^*)R$  to the  $1 - \lambda_1$  late consumers so each receives a payoff of

$$\beta_{21} = \frac{(1 - y^*)R}{1 - \lambda_1} = c_{21}^*.$$

This demonstrates that the banks can implement the constrained efficient allocation given the open market operations of the central bank described. As in Section 5.1 we need to show that it is individually optimal for each bank to implement this given the central bank is pursuing its optimal policy. Given  $P = 1$  it follows that the choice of  $y$  does not matter at the individual bank level as before and the aggregate level of  $y$  is determined by market clearing. Each bank's problem is then

$$\text{Max}_d \pi [\lambda_0 u(d) + (1 - \lambda_0) u \left( \frac{(1 - \lambda_0 d)R}{1 - \lambda_0} + \gamma_{20} \right)] + (1 - \pi) [\lambda_1 u(d) + (1 - \lambda_1) u \left( \frac{(1 - \lambda_1 d)R}{1 - \lambda_1} \right)]$$

It can straightforwardly be checked that the objective function is a concave function of  $d$  so

that there is a unique optimal value of  $d$  for each bank. Moreover, this optimal level is  $d^*$ , the same as the value in the constrained efficient allocation.

## 6 Idiosyncratic and aggregate liquidity risk: $\eta > 0, \varepsilon > 0$

We continue to denote the constrained efficient allocation  $y^*, d^*$ , and  $c_{2\theta}^*$  as in (13)-(15). With both idiosyncratic and aggregate risk the open market operations of the central bank necessary to implement the constrained efficient allocation combine the elements from the two cases alone. At date 0 the government imposes a lump sum tax of  $X_0$  and gives it to the central bank. The central bank uses it to fund a portfolio of the short asset. At date 1 in state  $\theta = 0$  the central bank fixes the price of the long asset at  $P_0 = 1$  by removing liquidity from the market. In order to do this it uses government securities that pay  $R$  at date 2 and sells them at  $P_0 = 1$ . The quantity of government securities issued at date 1 is denoted  $X_1$ . In order to ensure the price of the long asset can be successfully fixed, it is necessary that

$$X_0 + X_1 = \lambda_1 d^* - \lambda_0 d^* = \varepsilon d^*. \quad (16)$$

This ensures that all of the excess liquidity is drained from the banks into the central bank and there is no pressure to push up  $P_0$  in state  $\theta = 0$ .

The date 2 interest paid on the securities issued at date 1 is paid from the short asset held by the central bank. If any is left over then this is paid out as a lump sum grant to late consumers. If the resources of the central bank are insufficient then the shortfall is covered by a lump sum tax. In state  $\theta = 1$  the central bank needs to supply liquidity to the market because there is just enough liquidity in the financial system  $y^* = \lambda_1 d^*$  to satisfy the aggregate demand in state  $\theta = 1$ . If the central bank did not release this liquidity the banks would not have enough to satisfy the demands of their early consumers. It does this by using the short asset it holds to buy the long asset. This enables it to fix the price at  $P_1 = 1$ .

We next determine the choice of  $X_0$  and  $X_1$  and the banks' portfolio that implements the constrained efficient allocation. At date 0 after the lump sum tax of  $X_0$  the depositors have  $1 - X_0$  remaining and they deposit this in the banks. The banks choose a portfolio of  $y^* - X_0$  in the short asset and  $1 - y^*$  in the long asset.

At date 1 in state  $\theta = 0$  the  $i$  banks need liquidity  $\lambda_{0i}d^*$  to satisfy the demands of their early consumers. They have  $y^* - X_0$ . They therefore sell  $\lambda_{0i}d^* - (y^* - X_0)$  of the long asset. (Note that if  $\lambda_{0i}d^* < (y^* - X_0)$  this is negative and they are buying the long asset, the total supply of which includes that issued by the central bank.) The amount of the long asset they have remaining is  $1 - y^* - [\lambda_{0i}d^* - (y^* - X_0)] = 1 - \lambda_{0i}d^* - X_0$ . At date 2 they are able to use the payoffs of these long term assets to give each of their  $1 - \lambda_{0i}$  late consumers to provide a payout of

$$\beta_{20i} = \frac{(1 - \lambda_{0i}d^* - X_0)R}{1 - \lambda_{0i}}.$$

In addition to this payoff from the bank the late consumers receive a lump sum grant (or tax) from the government. The central bank has  $X_0$  in cash from date 0. As explained above, at date 1 they issue  $X_1 = \varepsilon d^* - X_0$  of securities that pay  $R$  at date 2. Thus at date 2 the total amount owed in interest is  $X_1R = (\varepsilon d^* - X_0)R$ . The central bank holds the proceeds of the debt issue  $X_1 = \varepsilon d^* - X_0$  in the short asset. In total they have  $X_0 + \varepsilon d^* - X_0 = \varepsilon d^*$  of the short asset. This allows a lump sum grant to each of the  $1 - \lambda_0$  late consumers of

$$\gamma_{20} = \frac{\varepsilon d^* - (\varepsilon d^* - X_0)R}{1 - \lambda_0} = \frac{X_0R - \varepsilon d^*(R - 1)}{1 - \lambda_0}.$$

The amount received by each of the late consumers in the  $i = H, L$  banks is

$$\beta_{20i} + \gamma_{20} = \frac{(1 - \lambda_{0i}d^* - X_0)R}{1 - \lambda_{0i}} + \frac{X_0R - \varepsilon d^*(R - 1)}{1 - \lambda_0}. \quad (17)$$

In order to implement the constrained efficient allocation, it is necessary that this is equal

to the constrained efficient allocation  $c_{20}^*$  so we have

$$\beta_{20i} + \gamma_{20} = \frac{(1 - \lambda_{0i}d^* - X_0)R}{1 - \lambda_{0i}} + \frac{X_0R - \varepsilon d^*(R - 1)}{1 - \lambda_0} = c_{20}^* = \frac{\varepsilon d^* + (1 - y^*)R}{1 - \lambda_0},$$

using (14).

As with just idiosyncratic risk it can be seen

$$X_0 = 1 - d^* \tag{18}$$

allows both  $H$  and  $L$  banks to implement the constrained efficient allocation.

It remains to show that  $X_0 = 1 - d^*$  allows the banks to ensure early consumers receive  $d^*$  and late consumers receive  $c_{21}^*$ . Similarly to (17) it can be shown that late consumers receive

$$\beta_{21\theta} + \gamma_{21} = \frac{(1 - \lambda_{1\theta}d^* - X_0)R}{1 - \lambda_{1\theta}} + \frac{X_0R}{1 - \lambda_1}. \tag{19}$$

The main difference here is in last term, which is the lump sum grant. As explained above, in state  $\theta = 1$  the central bank at date 1 uses the short term asset to purchase  $X_0$  of the long asset. This pays off a total of  $X_0R$  at date 2 to be distributed among the  $1 - \lambda_1$  late consumers.

Again substituting  $X_0 = 1 - d^*$  and using  $y^* = \lambda_1 d^*$  it follows that

$$\beta_{21\theta} + \gamma_{21} = \frac{(1 - y^*)R}{1 - \lambda_1} = c_{21}^*.$$

Thus the central bank policy described allows banks to implement the constrained efficient allocation.

It remains to show that it is optimal for each individual bank to choose  $d = d^*$  as the optimal contract. This can be done in the same way as in Sections 5.1 and 5.2.

Just as in Section 4, if aggregate uncertainty is sufficiently large relative to idiosyncratic uncertainty the banks will stop trading with each other in state  $\theta = 0$  and in this sense

the market freezes. As there, the condition for the market to freeze is that  $\lambda_1 d > \lambda_0 H d$  or equivalently

$$\varepsilon > \eta.$$

The difference here is that the  $H$  banks continue to trade with the central bank, however. The central bank sells long securities to the banks but that is the only trade that takes place. Since now these allocations with market freezes are constrained efficient they cannot be improved on. Thus the observation that banks stop lending to each other does not necessarily mean there is a market failure or inefficiency. The model here provides an example where market freezes are efficient.

## 7 Complete markets

In the model analyzed so far, markets are incomplete because it is not possible to hedge aggregate or idiosyncratic liquidity risk. In this section we consider the allocation that would occur with complete markets where liquidity risk can be hedged. This version of the model is a special case of that considered in Allen and Gale (2004a). They show that with complete markets and incomplete contracts of the type considered here the allocation is constrained efficient. In other words, a planner subject to the constraint of using a fixed payment in the first period cannot improve upon the complete markets allocation. Institutionally there are a number of ways that complete markets can be implemented. We focus on two. The first is where all trades occur at date 0. This is termed the *static* case. The second is where trades occur at both dates 0 and 1 and this is termed the *dynamic* case.

### 7.1 Static complete markets

One of the simplest institutional structures to understand is where all trade takes place at date 0. Initially we will focus on aggregate risk and will introduce idiosyncratic risk at a later stage. For the moment,  $\lambda_0 = \bar{\alpha}$  and  $\lambda_1 = \bar{\alpha} + \varepsilon$ .

So far we have assumed that assets are held by the bank. Since the assets are produced with constant returns to scale, with complete markets there will be zero profits associated with producing them. Therefore it does not matter which agents hold them. Let's suppose initially firms hold them and issue securities against them. Banks use the funds from deposits to buy these securities. We will model these securities in the form of Arrow securities where each security pays off 1 in a particular state and nothing in any of the other states. All of these Arrow securities are traded at date 0.

There are five aggregate states in total. At date 0 there is one state. There are two states  $\theta = 0, 1$  at the two subsequent dates  $t = 1, 2$  to give four further states  $(t, \theta)$  for a total of five. We take consumption at date 0 as the numeraire with the price of the Arrow security paying off 1 unit of consumption in that state normalized at 1. The prices of the Arrow securities that pay off one unit of consumption in the other states  $(t, \theta)$  are denoted  $p_{t\theta}$ .

We can represent the short assets and the long asset by their payoffs in the five states  $(0, 10, 11, 20, 21)$  as follows:

Asset	Payoffs	Zero-profit condition
Short asset from date 0 to 1	$(-1, 1, 1, 0, 0)$	$-1 + p_{10} + p_{11} \leq 0$
" " " " 1 to 2 in state $\theta = 0$	$(0, -1, 0, 1, 0)$	$-p_{10} + p_{20} \leq 0$
" " " " 1 to 2 in state $\theta = 1$	$(0, 0, -1, 0, 1)$	$-p_{11} + p_{21} \leq 0$
Long asset from date 0 to 1	$(-1, 0, 0, 1, 1)$	$-1 + p_{20}R + p_{21}R \leq 0$

If the zero profit condition is satisfied with an equality the asset is produced. If it is satisfied with a strict inequality it is not produced. To implement the constrained efficient allocation for Example 1 we have

$$-1 + p_{10} + p_{11} = 0; -p_{10} + p_{20} = 0; -p_{11} + p_{21} < 0; -1 + p_{20}R + p_{21}R = 0.$$

The problem of the representative bank is to use the Arrow security markets at date 0 to purchase the units of consumption to maximize its depositors expected utility. The total

amount of consumption it purchases is  $\lambda_\theta d$  at date 1 and  $(1 - \lambda_\theta)c_{2\theta}$  at date 2 for  $\theta = 0, 1$ . The bank chooses  $d, c_{20}$ , and  $c_{21}$  to

$$\begin{aligned} \max \quad & \pi [\lambda_0 u(d) + (1 - \lambda_0)u(c_{20})] + (1 - \pi) [\lambda_1 u(d) + (1 - \lambda_1)u(c_{21})] \\ \text{s.t.} \quad & p_{10}\lambda_0 d + p_{20}(1 - \lambda_0)c_{20} + p_{11}\lambda_1 d + p_{21}(1 - \lambda_1)c_{21} = 1 \\ & 0 \leq d, c_{20}, c_{21}. \end{aligned} \tag{20}$$

The first line is the expected utility of the depositor. The second is the budget constraint in the date 0 markets. There is a single budget constraint because all transactions take place at date 0. The third line has the usual non-negativity constraints.

Denoting the Lagrange multiplier for the budget constraint  $\mu$ , the first order conditions for the choice of  $d, c_{20}$ , and  $c_{21}$  are:

$$\pi \lambda_0 u'(d) + (1 - \pi) \lambda_1 u'(d) + \mu(p_{10}\lambda_0 + p_{11}\lambda_1) = 0$$

$$\pi u'(c_{20}) + \mu p_{20} = 0$$

$$(1 - \pi) u'(c_{21}) + \mu p_{21} = 0.$$

Substituting the constrained efficient values of  $d, c_{20}$ , and  $c_{21}$  into these, and using the budget constraint and the zero profit conditions, it is possible to derive the prices that implement the constrained efficient allocation. These prices allow the firms to produce the assets at zero profits, and the banks to maximize the depositors' welfare.

So far we have abstracted from idiosyncratic risk. We next consider how this can be accommodated. Suppose each firm issues state-contingent Arrow securities based on the shock  $H$  or  $L$  experienced by the purchasing bank. They issue these securities in small amounts and to enough banks that the idiosyncratic risk is diversified away.

Each bank will buy enough of the  $H$  and  $L$  securities to cover their needs in each of the states. As usual we denote  $\lambda_{\theta i} = \alpha_i + \varepsilon\theta$  for  $\theta = 0, 1$  and  $i = H, L$ . The Arrow securities each bank buys are  $\lambda_{\theta i}d$  at date 1 and  $(1 - \lambda_{\theta i})c_{2\theta}$  at date 2 for  $\theta = 0, 1$  and  $i = H, L$ . The price

of these securities are  $p_{t\theta i}$  for  $t = 1, 2, \theta = 0, 1$  and  $i = H, L$ .

In order for the banks to be able to afford the optimal state contingent securities it is necessary that

$$\lambda_{\theta H} p_{t\theta H} + \lambda_{\theta L} p_{t\theta L} = \lambda_{\theta} p_{t\theta} \text{ for } t = 1, 2 \text{ and } \theta = 0, 1.$$

Since the aggregate state  $t\theta$  is the same for each  $H$  and  $L$ , and 0.5 of the banks are  $H$  and 0.5 are  $L$  consider the symmetric equilibrium with

$$p_{t\theta H} = p_{t\theta L} = 0.5p_{t\theta}.$$

This ensures that the banks can afford to purchase the constrained efficient allocation. Since this gives the highest expected utility for the depositors, it is the best that the banks can do.

In the case of incomplete markets, the banks held the assets. With complete markets we have, for simplicity, been assuming that firms hold the assets and issue the securities. Since there are zero profits from producing the assets we could just as well assume that the banks held the assets. In order to obtain the benefits of diversification, they would issue securities against the assets in the same way as the firms. They would also buy them in the same way as previously. Thus they would be on both sides of the market buying and selling large numbers of securities.

## 7.2 Dynamic complete markets

The institutional structure where all trades take place at date 0 described above is only one institutional structure that will implement complete markets. Another structure is to have dynamic markets. As before, to see how these operate consider the case with no idiosyncratic risk first.

The market structure is as follows. The firms issue Arrow securities between dates 0 and 1. These are contingent on the state  $\theta = 0, 1$  and allow the banks to hedge this risk. For

simplicity, we assume that security  $0\theta$  pays off 1 unit of consumption at date 1 and has price  $p_{0\theta}^*$  at date 0. At date 1, there are markets for date 2 consumption that the banks and firms can also trade in. Security  $1\theta$  pays off 1 unit of consumption at date 2 and sells for price  $p_{1\theta}^*$  at date 1.

In the dynamic market buying 1 unit of date 1 consumption is the same as buying it in the static market so we must have

$$p_{0\theta}^* = p_{1\theta}.$$

To buy one unit of consumption at date 2 in state  $\theta$ , it is necessary to have  $p_{1\theta}^*$  at date 1. This is obtained by paying  $p_{0\theta}^*p_{1\theta}^*$  in the date 0 market for date 1 consumption in state  $\theta$ . Thus

$$p_{0\theta}^*p_{1\theta}^* = p_{2\theta}.$$

Also the banks can implement the constrained efficient allocation. Initially each bank buys  $z_\theta = \lambda_\theta d + p_{1\theta}^*(1 - \lambda_\theta)c_{2\theta}$  of consumption at date 1 for state  $\theta = 0, 1$ . This allows it to provide the  $d$  for its early consumers and buy the  $c_{2\theta}$  for its late consumers.

For dynamic markets we have so far ignored idiosyncratic risk. This can be accommodated, similarly to static markets, by having firms issue a large number securities that are contingent on the buying banks liquidity shock  $H$  or  $L$ . The banks buy  $z_{\theta i} = \lambda_{\theta i} d + p_{1\theta}^*(1 - \lambda_{\theta i})c_{2\theta}$  of date 1 consumption in states  $\theta = 0, 1$  and then use whatever is left over after paying their early consumers to buy date 2 consumption for the late consumers. As with static markets the price of the securities are such that

$$p_{0\theta H}^* = p_{0\theta L}^* = 0.5p_{0\theta}^*.$$

With these prices the bank can afford to purchase the constrained efficient allocation, and the firms satisfy the zero profit conditions.

To sum up, in this section we have provided two institutional structures that implement complete markets. Both require a large number of markets and securities to allow the

idiosyncratic risk to be diversified away. Clearly it is not the case that all trades are made at a single date in practice. In that sense the dynamic markets are in some sense more realistic. However, they still require that banks buy and sell a large number of securities. In practice, issuing securities is costly and this makes even the dynamic markets case implausible. This is why the role of the central bank in implementing the constrained efficient allocation is so important.

## 8 Concluding remarks

This paper has developed a simple model of the interbank market. We have shown how central bank intervention in the interbank market can improve welfare in a variety of situations.

The model is very simple. However, it can be extended in a number of directions to consider important issues. First, so far we have ignored bankruptcy of financial institutions. Incorporating this will allow open market operations to be compared with lender of last resort policies. Second, the model is a real one in that all the funds of the bank that are used for intervention are raised through lump sum taxes. If the bank uses seigniorage instead then this should allow some insight into the relationship between monetary policy and financial stability. Third, in the model the interest rates are tied down by technologies. One important extension is to change the model so that the short term rate is not determined by the return of the technology. This change would give the central bank the ability to vary the rate more than in the current model and permits a richer analysis of central bank intervention.

Another important issue is how monetary policy affects real activity. Here we have banks choosing portfolios of assets. These can be thought of as loans. By introducing loans to firm explicitly and seeing how banks' decisions interact with firms' decisions we can obtain some insight into the relationship between monetary policy and economic activity.

## References

- Acharya V., D. Gromb and T. Yorulmazer (2008). “Imperfect Competition in the Inter-Bank Market for Liquidity as a Rationale for Central Banking,” working paper, London Business School.
- Acharya V. and T. Yorulmazer (2008). “Cash-in-the-Market Pricing and Optimal Resolution of Bank Failures,” *Review of Financial Studies*, forthcoming, .
- Allen, F. and E. Carletti (2006). “Credit Risk Transfer and Contagion,” *Journal of Monetary Economics* 53, 89-111.
- and — (2008). “Mark-to-Market Accounting and Liquidity Pricing,” *Journal of Accounting and Economics*, 45, 358-378.
- Allen, F. and D. Gale (2000). “Financial Contagion,” *Journal of Political Economy* 108, 1-33.
- and — (2004a). “Financial Intermediaries and Markets,” *Econometrica* 72, 1023-1061.
- and — (2004b). “Financial Fragility, Liquidity, and Asset Prices” *Journal of the European Economic Association* 2, 1015-1048.
- and — (2007). *Understanding Financial Crises*, New York and Oxford: Oxford University Press, Clarendon Lectures in Finance.
- Bhattacharya S. and D. Gale (1987). “Preference Shocks, Liquidity and Central Bank Policy,” in W. Barnett and K. Singleton (eds.), *New Approaches to Monetary Economics*, Cambridge University Press, 69-88.
- Diamond D. and R. Rajan (2005). “Liquidity Shortages and Banking Crises”, *Journal of Finance*, 60, 615-647.
- Flannery M. (1996). “Financial Crises, Payment System Problems and Discount Window Lending,” *Journal of Money, Credit and Banking*, 28, 804-824.
- Freixas X., B. Parigi and J.C. Rochet (2004). “The Lender of Last Resort: A 21th Century Approach”, *Journal of the European Economic Association*, 2, 1085-1115.
- Freixas X. and J. Jorge (2007). “The Role of Interbank Markets in Monetary Policy: A

Model with Rationing”, working paper, Universitat Pompeu Fabra.

Goodfriend M. and R. King (1988). ”Financial Deregulation, Monetary Policy and Central Banking”, *Federal Reserve Bank of Richmond Economic Review*, 74, 3-22.

Gorton H. and L. Huang (2004). “Liquidity, Efficiency and Bank Bailouts”, *American Economic Review* 94, 455-483.

— and — (2006). “Banking Panics and Endogeneity Coalition Formation”, *Journal of Monetary Economics* 53, 1613-1629.

Heider, F., M. Hoerova and C. Holthausen (2008). “Information Asymmetries in the Interbank Market: Theory and Policy Response,” mimeo, European Central Bank, Frankfurt.

Holmstrom, B. and J. Tirole (1998). “Private and Public Supply of Liquidity,” *Journal of Political Economy* 106, 1-40.

Repullo R. (2005). “Liquidity, Risk Taking, and the Lender of Last Resort,” *International Journal of Central Banking* 1, 47-80.