ESTIMATION OF DSGE MODELS WHEN THE DATA ARE PERSISTENT

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Abstract

An active area of research in macroeconomics is to take DSGE models to the data. Much of the focus has been on estimation and testing of models solved under specific assumptions about how the exogenous variables grow over time. In this paper, we first show that if the trends assumed for the model are incompatible with the observed data, or that the detrended data used in estimation are inconsistent with the stationarity concepts of the model, the estimates can be severely biased even in large samples. Estimates of parameters governing transmission mechanisms can be dramatically distorted. We then consider the QD (quasi-differenced), the unconstrained first difference (Δ DT), and the HP-HP estimators. All three are robust to whether shocks in the model are assumed to be permanent or transitory. Root-T consistent and asymptotically normal estimates can be obtained. The estimators do not require the researcher to take a stand on the dynamic properties of the data, but simulations show that they work as well as when the stationarity property of the shock process is correctly imposed. Importantly, it is far more accurate than standard estimators when the model parameter is near the unit circle. These properties hold even when there are multiple persistent shocks.

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are now accepted as the primary framework for macroeconomic analysis. Until recently, counterfactual experiments were conducted by assigning the parameters of the models with values that are loosely calibrated to the data. More recently, serious efforts have been made to estimate the model parameters using classical and Bayesian methods. This permits researchers to assess how well the models fit the data both in and out of samples. Formal estimation also permits errors arising from sampling or model uncertainty to be explicitly accounted for in counterfactual policy simulations. Arguably, DSGE models are now taken more seriously as a tool for policy analysis because of such serious econometric investigations.

As is well known, economic data are highly persistent and possibly non-stationary. It is common practice to allow shocks in DSGE models to have persistent effects. When one or more forcing processes in a DSGE model are non-stationary, the model variables in level form have to be first normalized by appropriate trending variables. The variables in the log-linearized model are then interpreted as deviations from the steady state or balanced growth path. In order to take the model to the data, a researcher must construct data analogs of the model concepts, and in doing so, must choose a method for detrending the data. This paper points out two potential problems specific to the estimation of DSGE models when either the data and/or the model variables are persistent or non-stationary. The first problem arises when the method of detrending does not agree with the definition of the trends in the model. The second problem arises when the data are detrended to match the model concepts but that the empirically detrended data remain non-stationary or are over-differenced. Both issues can pose problems for estimation and inference. Hereafter, we refer to these issues as Data Detrending (DD) and Model Trend Specification (MTS) problems. A concise overview of the issues associated with estimating DSGE models is as follows:

	Step 1		Step 2		Step 3
	Model Specification	→	Data Detrending	→	Estimation
Problems:		MTS		DD	

Problem (DD) is concerned with how the observed data are filtered. The filtered data can be stationary and yet the trends associated with the stationary component of the data can be inconsistent with how the trends are defined in the model. Problem (DD) would most likely arise when a researcher detrends the data to ensure that the deviation from a trend component is stationary, but is unaware that the trends that accomplish this task are inconsistent with the trends specified in the model.

For example, the model may specify the trend as a random walk, but the data may be detrended by a two-sided symmetric filter. Whereas the stationary component in the model is white noise, the filtered series can be serially correlated. In this case, the error term associated with the empirical Euler equations can be serially correlated. The moment conditions used to estimate the parameters will not be zero even in the population.

Problem (MTS) is concerned with whether the assumption about the trend in the model is consistent with the trend in the data. Problem (MTS) is often related to whether the detrended data are stationary. This issue can arise if, for example, the model assumes that technology is trend stationary and thus the data are linearly detrended accordingly. However, the detrended data will still be non-stationary if in fact the data contain stochastic trends. As is well known, classical inference procedures can be misleading when the regressors are non-stationary or highly persistent, and estimation of a spurious regression cannot be ruled out. An additional issue that confronts researchers is that in finite samples, it is very difficult to ascertain whether the data are stationary or not. Yet, existing estimators of DSGE models require that the researcher takes a stand on the stationarity property of the data.

An error in either model trend specification or data detrending can seriously distort the results in the estimation step. This problem is highly important because, as Cogley (2001) observes, "The RBC methodology is motivated by a desire to formulate estimators and tests that are not too distorted by trend misspecification." However, it is generally impossible to separate a cycle from an exogenous trend using atheoretical filters because such procedures do not incorporate information about the economic model.[1] As a result, there can be large discrepancies between the properties of the cyclical component in the model and in the data and hence estimates of the structural parameters can be badly biased. On the other hand, explicit specification of the trend involves (MTS) and (DD) problems.

Table 1 is a non-exhaustive listing of how trends are treated in some notable papers. While there are exceptions, the majority of the analysis assumes that non-stationarity in the models is due to a deterministic trend. The empirical analysis then proceeds to estimate the models on linearly detrended data. Stochastic trends are assumed in some studies and the first differenced data are then used in estimation. But since the seminal work of Nelson and Plosser (1982), there has been ongoing debate whether the trend or difference stationary is a better characterization of macroeconomic variables. While much is known about estimation and inference of linear models with non-stationary data, little is known about how the treatment of trends affects estimation of DSGE models. This paper sheds some light on this issue.

[1] Obviously, the separation of trend and cycle may be even more problematic for general equilibrium models with endogenous growth and models without balanced growth path.

This paper is intimately related to previous literature investigating properties of the filtered data. From this literature we know that improper filtering can alter the persistence and the volatility of the series (e.g., Cogley and Nason (1995)), induce spurious correlations in the filtered data (e.g., Harvey and Jaeger (1993)), change error structure (e.g., Singleton (1988)), distort inference (e.g., Christiano and den Haan (1996)) or even yield non-stationary series (e.g., Nelson and Kang (1981)). However, much of this literature is focused on univariate analysis and relatively little is known about the effects of filtering on the estimates of the structural parameters in DSGE models. The systems approach provides a complete characterization of the model and thus the estimates are more efficient if the model is correctly specified. But misspecification in one equation can affect estimates in other equations. In an early contribution, King and Rebelo (1993) simulate an RBC model and show that HP (Hodrick-Prescott) filtered data are qualitatively different from the raw data. Although these authors do not estimate the model on filtered data, they hint that the estimates of the structural parameters can be adversely affected by filtering. Fukac and Pagan (2006) also consider how the treatment of trends might affect estimation of DSGE models but the analysis is also confined to a univariate framework.

In a similar vein, analysis of Problem (MTS) has been generally conducted within the univariate framework. In a study closely related to ours, Cogley (2001) investigates formally how Problem (MTS) can affect the estimates of structural parameters. He shows that inappropriate choice of trend (i.e., trend stationary versus difference stationarity forcing variables) can lead to strong biases in the parameter estimates. He considers several possibilities to circumvent Problem (MTS) and finds that using cointegration relationships in unconditional Euler equations works the best since in this formulation, moments used in GMM estimation remain stationary irrespective of whether the data are trend or difference stationary. Our approach based on estimating covariance structures is different from and complementary to Cogley's approach.[2] Instead of comparing estimators, we study the properties of the covariance structure estimator alone to focus on the sensitivity of the estimates to the model underlying the covariance structure as well as to the choice of sample analogues of the model variables.[3] As we will see, our method is also more general.

Specifically, we propose a robust strategy to handle uncertainty as to whether the data are trend or difference stationary. The proposed approach applies the same transformation (filter) to both data and model series such that transformed series are stationary. We illustrate this approach with three

[2]Combination of these two approaches is straightforward and, although we do not investigate the benefits on combining cointegration and, for example, quasi-differencing formally, one may expect that such a combination can be quite fruitful in sharpening the estimates.

[3]Gregory and Smith (1996) investigate this problem from another perspective. Using a *calibrated* business cycle model, they try to find a trend component that can produce a cyclical component in the data similar to the cyclical component in the model.

transformations: quasi-differencing, first differencing, and HP filter. The approach is shown to be effective even when there are multiple shocks, a subset of which may be permanent. Although our analysis is motivated as classical estimator, it can be adapted into a Bayesian framework.

We use a basic stochastic growth model to illustrate the problems under consideration. When the trends assumed for the model agree with the trends present in the data *and* the same filter is applied to model and data series, the estimated parameters are mean and median unbiased. Otherwise, the estimates can deviate significantly from the true values. Estimates of parameters governing the propagation and amplification mechanisms in the model can be greatly distorted or poorly identified.

The structure of the paper is as follows. In the next section, we lay out a standard neoclassical growth model. We linearize the model and show how one can solve it under different assumptions about trends in the forcing variables. We present the estimation procedure and illustrate Problems (DD) and (MTS) with a few specific examples. In Section 3, we report simulation results. In particular, we demonstrate a superior performance of our robust approach and distortions in popular estimators due to Problems (DD) and (MTS). We consider several extensions of the baseline growth model to highlight the issues associated with estimation of endogenous propagation/amplification mechanisms and estimation of models with multiple structural shocks. We also briefly contrast statistical properties of our approach and popular alternatives. In Section 4, we develop a general framework for using quasi-differencing to estimate structural parameters in linearized DSGE models. We conclude in Section 5.

2 An Example: Neoclassical Growth Model

2.1 The General Setup

We use the stochastic growth model to highlight the problems under investigation. The problem facing the central planner is:

$$\max E_t \sum_{t=0}^{\infty} \beta^t \Big(\ln C_t - \theta(L_t/Q_t) \Big)$$

subject to

$$\begin{aligned}
Y_t &= C_t + I_t = K_{t-1}^{\alpha}(Z_t L_t)^{(1-\alpha)} \\
K_t &= (1-\delta)K_{t-1} + I_t \\
Z_t &= \exp(\bar{g}t)\exp(u_t^z), \quad u_t^z = \rho_z u_{t-1}^z + e_t^z, \quad |\rho_z| \le 1 \\
Q_t &= \exp(u_t^q), \qquad\qquad u_t^q = \rho_q u_{t-1}^q + e_t^q, \quad |\rho_q| \le 1.
\end{aligned}$$

where Y_t is output, C_t is consumption, K_t is capital, L_t is labor input, Z_t is the level of technology, and Q_t is a labor supply shock. We allow ρ_q and ρ_z to be on the unit circle. The first order conditions are:

$$
\begin{aligned}
\theta C_t &= (1-\alpha)K_{t-1}^{\alpha} Z_t^{(1-\alpha)} L_t^{-\alpha} Q_t \\
E_t C_{t+1} &= \beta C_t \left(\alpha K_t^{\alpha-1}(Z_{t+1}L_{t+1})^{(1-\alpha)} + (1-\delta)\right) \\
K_{t-1}^{\alpha}(Z_t L_t)^{(1-\alpha)} &= C_t + K_t - (1-\delta)K_{t-1}
\end{aligned}
$$

If $\bar{g} = 0$ and $|\rho_z|, |\rho_q| < 1$, then under regularity conditions, a solution for model log-linearized around the steady state values exists. But once technology is allowed to grow over time, the model solution as well as the estimation approach depends on the properties of Z_t and Q_t.

Let lower case letters denote the natural logarithm of the variables, e.g. $c_t = \log C_t$. Let c_t^*, be such that $c_t - c_t^*$ is stationary; k_t^* and z_t^* are similarly defined. Note that c_t^* and k_t^* are model concepts. Hereafter, we will use DT and ST to refer to the case when $|\rho_z| < 1$ and $|\rho_z| = 1$, respectively. The assumption on $|\rho_q|$ will vary depending on the context. Where appropriate, we will drop Q_t to simplify the analysis.

2.2 Solving the One Shock Model

To fix ideas, suppose for now that technology is the only shock in the system. Hence, Q_t is suppressed. We consider separately when $|\rho_z| < 1$ and when $|\rho_z| = 1$.

When $|\rho_z| < 1$, $c_t^* = k_t^* = \bar{g}t$. The detrended variables in the model are then defined as $\widehat{c}_t = c_t - c_t^*$, $\widehat{k}_t = k_t - k_t^*$ and $\widehat{l}_t = l_t$.[4] The log-linearized model in terms of $\widehat{c}_t, \widehat{k}_t, \widehat{l}_t$ is

DT Model

$$
\begin{aligned}
E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \\ \widehat{l}_{t+1} \end{bmatrix} &= \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha-1 \end{bmatrix} \begin{bmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \\ \widehat{l}_{t} \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \widehat{c}_{t-1} \\ \widehat{k}_{t-1} \\ \widehat{l}_{t-1} \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ -A_0 \\ 0 \end{bmatrix} u_{t+1}^z + \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha-1 \end{bmatrix} u_t^z \qquad (1)
\end{aligned}
$$

where we suppress constant terms and define entries of the matrices in (1) as follows

$$
\begin{aligned}
A_0^* &= 1 - \beta\frac{1-\delta}{1+\bar{g}}, \quad A_0 = (\alpha-1)A_0^*, \quad A_4 = -\alpha - (1-\delta)A_3, \\
A_3 &= \frac{\alpha\beta}{(1+\bar{g})A_0^*}, \quad A_2 = (1+\bar{g})A_3, \qquad 1 = A_1 + A_2 - (1-\delta)A_3.
\end{aligned}
$$

[4]Note that labor L_t is stationary for all $|\rho_z| \leq 1$ and thus we do not need to scale it.

We will refer to (1) as the trend stationary (DT) representation of the model. Let $\widehat{m}_t = (\widehat{c}_t, \widehat{k}_t, \widehat{l}_t)'$. Since a shock to technology has temporary effects, $\widehat{m}_t$ is stationary. We can compactly write (1) as

$$E_t \Gamma_2^D \widehat{m}_{t+1} = \Gamma_0^D \widehat{m}_t + \Gamma_1^D \widehat{m}_{t-1} + \Psi_1^D u_{t+1}^z + \Psi_0^D u_t^z.$$

The QZ decomposition or similar methods can be used to solve the system of expectation equations for the reduced form. Denote this solution by

$$\begin{pmatrix} \widehat{m}_t \\ u_t^z \end{pmatrix} = \Pi_{DT} \begin{pmatrix} m_{t-1} \\ u_{t-1}^z \end{pmatrix} + B_{DT} e_t^z.$$

ST Model When $|\rho_z| = 1$, the log-linearized model is expressed in $\widetilde{c}_t$, $\widetilde{k}_t$, and $\widetilde{l}_t$, where

$$\widetilde{c}_t = c_t - z_t, \quad \widetilde{k}_t = k_t - z_t, \quad \widetilde{l}_t = l_t.$$

The model is now represented by the following system of expectational equations:

$$E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t+1} \\ \widetilde{k}_{t+1} \\ \widetilde{l}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha - 1 \end{bmatrix} \begin{bmatrix} \widetilde{c}_t \\ \widetilde{k}_t \\ \widetilde{l}_t \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t-1} \\ \widetilde{k}_{t-1} \\ \widetilde{l}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ -A_0 \\ 0 \end{bmatrix} e_{t+1}^z + \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha - 1 \end{bmatrix} e_t^z \tag{2}$$

We will refer to (2) as the stochastic trend (ST) representation of the model. Let $\widetilde{m}_t = (\widetilde{c}_t, \widetilde{k}_t, \widetilde{l}_t)'$ and compactly write the system as

$$E_t \Gamma_2^S \widetilde{m}_{t+1} = \Gamma_0^S \widetilde{m}_t + \Gamma_1^S \widetilde{m}_{t-1} + \Psi_1^S e_{t+1}^z + \Psi_0^S e_t^z.$$

The solution to the model is

$$\widetilde{m}_t = \Pi_{ST} \widetilde{m}_{t-1} + B_{ST} e_t^z.$$

Now $\widetilde{m}_t$ and $\widehat{m}_t$ are related as follows:

$$\widetilde{c}_t = \widehat{c}_t - u_t^z, \quad \widetilde{k}_t = \widehat{k}_t - u_t^z, \quad \widetilde{l}_t = \widehat{l}_t.$$

Effectively, subtracting u_t^z from appropriate variables as in the ST model changes the object of interest in the model from $\widehat{m}_t$ (which would not be stationary under ST) to $\widetilde{m}_t$ (which is stationary under ST).

$\Delta^1 DT$ **Model** An alternative to solving the ST model when $|\rho_z| = 1$ is to consider the first-differenced representation of the DT model.

$$E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta^1 \widehat{c}_{t+1} \\ \Delta^1 \widehat{k}_{t+1} \\ \Delta^1 \widehat{l}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha - 1 \end{bmatrix} \begin{bmatrix} \Delta^1 \widehat{c}_{t} \\ \Delta^1 \widehat{k}_{t} \\ \Delta^1 \widehat{l}_{t} \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \Delta^1 \widehat{c}_{t-1} \\ \Delta^1 \widehat{k}_{t-1} \\ \Delta^1 \widehat{l}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ -A_0 \\ 0 \end{bmatrix} e^z_{t+1} + \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha - 1 \end{bmatrix} e^z_t \quad (3)$$

Here the superscript "1" in Δ^1 emphasizes that ρ_z is constrained to be equal to one. The solution is given by

$$\Delta^1 m_t = \Pi_{\Delta^1} \Delta^1 m_{t-1} + B_{\Delta^1} e^z_t.$$

Clearly, first differencing removes the permanent shock in $\widehat{m}_t$, while $\widetilde{m}_t$ subtracts the permanent shock from $\widehat{m}_t$. Not surprisingly, (2) and (3) both yield stationary solutions to the ST model.[5]

The system of equations (1), (2) and (3) both correspond to the same stochastic growth model. As one would expect, the rational expectations solution for variables in *levels* is the same irrespective of which model we solve. The models are distinguished only in what variables we analyze, i.e., $\widehat{m}_t$ for DT, $\Delta\widehat{m}_t$ for Δ^1DT, and $\widetilde{m}_t$ for ST. However, it is these *normalized* variables that are typically compared to the filtered data. The distinction between $\widetilde{m}_t$, $\Delta\widehat{m}_t$, and $\widehat{m}_t$ is important because the former two are stationary when $\rho_z = 1$ while $\widehat{m}_t$ is not. Although one can arrive at the ST system by solving the DT model and re-defining variables if $\rho_z = 1$, one should not use data analogue of $\widehat{m}_t$ in estimation because $\widehat{m}_t$ is a vector of non-stationary model variables with $\rho_z = 1$. Recall that classical estimation assumes that the data are stationary and, thus, estimation requires stationary data analogues of the model concepts. On the other hand, $\widetilde{m}_t$ and $\Delta\widehat{m}_t$ are stationary when $\rho_z = 1$ and thus are model concepts suitable for estimation.

2.3 Filtering the Data

To take the model to the data, one needs stationary data analogue of the model concepts. Suppose we observe the data for $d_t = (c_t, k_t, l_t)$. Let $d^c_t = (c^c_t, k^c_t, l^c_t) = (c_t - c^\tau_t,\ k_t - k^\tau_t,\ l_t - l^\tau_t)$ denote the data filtered to become stationary. We consider three possibilities.

[5] One may also exploit cointegration relationships to construct stationary linear combinations of the non-stationary variables. For example, $c_t - y_t$ is stationary for all $|\rho_z| \leq 1$. That is, the cointegration vector (-1, 1) nullifies the deterministic and stochastic trends in c_t and y_t, if they exist. Although in our basic model using this vector in estimation does not bring in new information since $c_t - y_t \propto l_t$, cointegration vectors (e.g., $y_t - k_t$) can enrich the model and make it more robust to problems that can arise when ρ_z approaches one. This approach is exploited in Cogley (2001) when he estimates Euler equations for cointegrated variables and variables in growth rates.

- Linear Trend (LT):

$$c_t^c = c_t - \bar{g}t, \quad k_t^c = k_t - \bar{g}t, \quad l_t^c = l_t. \tag{4}$$

- HP Trend (HP):

$$c_t^c = HP(L)c_t, \quad k_t^c = HP(L)k_t, \quad l_t^c = l_t. \tag{5}$$

- First Difference (FD):[6]

$$c_t^c = \Delta c_t - \bar{g}, \quad k_t^c = \Delta k_t - \bar{g}, \quad l_t^c = \Delta l_t. \tag{6}$$

Typically, linearly detrended data would replace the unobserved model variable $\widehat{m}_t$ when $|\rho_z| < 1$, while the HP filtered and first differenced data would stand in for $\widetilde{m}_t$ and $\Delta\widehat{m}_t$ when $\rho_z = 1$. HP filter can be and often is used in conjunction with $\widehat{m}_t$ when $|\rho_z| < 1$ because HP removes time trends as well. It is well known that the HP filter can alter the gain and phase of the cyclical components of the data (see e.g. Cogley and Nason (1995)) and change the error structure (see e.g. Singleton (1988)) in the univariate or single-equation framework. We examine formally how HP filter affects estimation in the DSGE context.

2.4 Estimation Procedure

Various non-Bayesian methods have been used to estimate the model as a system of equations. Two-step minimum distance approach (e.g., Sbordone (2006)), GMM/covariance structure (e.g., Christiano and den Haan (1996), Christiano and Eichenbaum (1992)), as well as simulation estimation (e.g., Altig et al. (2004)) can all be used. Ruge-Murcia (2005) provides a review of these methods. We use a method of moments estimator that minimizes the distance between data moments and model-implied moments. Our estimation procedure can be summarized as follows:

1: Compute $\widehat{\Omega}^d(0) = cov(d_t^c)$, the covariance matrix of the filtered series. Likewise, compute $\widehat{\Omega}^d(1)$, the first order sample auto-covariance.

2: Solve the rational expectations model (1), (2), or (3) for a guess of Θ, where Θ is the vector of structural parameters. Use the solution to analytically compute $\Omega^m(0)$ and $\Omega^m(1)$, the model implied covariance and autocovariance matrix for the model variables (which would be $\widehat{m}_t$, $\widetilde{m}_t$, or $\Delta\widehat{m}_t$).

[6] Even though the model predicts that labor is stationary, we first difference all series in the data because we solve the Δ^1DT model in first differences for all variables.

3: Let $\widehat{\omega}^d = (vech(\Omega^d(0))' \;\; vec(\Omega^d(1))')'$ and let $\omega^m(\Theta) = (vech(\Omega^m(0))' \;\; vec(\Omega^m(1))')'$. Estimate the structural parameters as $\widehat{\Theta} = \text{argmin}_{\Theta} \left\| \widehat{\omega}^d - \omega^m(\Theta) \right\|$.[7,8]

Before estimation, a researcher needs to take a stand on two issues. First, he/she must decide whether the model is solved in terms of $\widehat{m}_t$, which is stationary under DT but not ST, or $\widetilde{m}_t$, which is stationary under ST. In doing so, the researcher is also making an assumption whether the shocks in the model are permanent or transitory. Second, the researcher needs to map the model variables (which are stationary) to the observed data and must decide how to filter the data. Problem (DD) arises when the two steps are not mutually consistent. Problem (MTS) arises when the data analogue of the model variables are not stationary or are over-differenced; that is, the assumed trend in the model is different from the trend in the data.

To illustrate the problems, consider the following combinations of model variables and data filtering techniques:

True Model	Assumed Model and Variables	Filtering	Problems
1. DT	DT, $\widehat{m}_t$	LT	-
2. DT	DT, $\widehat{m}_t$	HP	(DD)
3. ST	ST, $\Delta^1\widehat{m}_t$	FD	-
4. ST	ST, $\widetilde{m}_t$	HP	(DD)
5. DT	ST, $\widetilde{m}_t$	HP	(DD),(MTS)
6. ST	DT, $\widehat{m}_t$	LT	(MTS)

Of the six configurations, (1) and (3) are correctly specified and the data are appropriately filtered. In both cases, the assumed trend is identical to the trend in the data and, thus, there is no Problem (MTS). Because the researcher applies the same filter to the model variables and the data series, Problem (DD) is not an issue. In case (2), the assumed trend in the model is consistent with the actual trend in the data (both are deterministic time trends) and there is no Problem (MTS). However, the HP filter applied to the data series has different properties than the filter in the model, which is the linear time trend. Since these two filters do not agree in general, the researcher faces Problem (DD). A similar problem arises in case (4). In case (6), the assumed trend and the choice of detrending technique are consistent; that is, the researcher applies an appropriate filter given his or her assumption about the trend. Hence, Problem (DD) does not apply for this case. On the other hand, because the researcher has to choose either DT or ST before estimation, his or her choice of DT is not consistent with the true data generating process (ST) and, consequently, Problem (MTS) applies to this case.

[7] One can use the values of the ratios or means contained in the constant terms as additional moments in estimation. However, since estimation of DSGE model is typically based on the second moments of the cyclical component of the data, we do not consider these additional moments in our analysis.

[8] We assume that shocks are homoscedastic. If shocks are heteroscedastic, one can consider additional higher moments to capture sources of heteroscedasticity.

Likewise, in case (5), the choice of the trend in the model (DT) does not agree with the trend in the data (ST). In addition, the choice of the filtering technique in the data is not consistent with the assumed trend in the model. It follows that Problem (MTS) is further complicated by Problem (DD).

Two observations can be made. First, Problem (DD) involves only inconsistency between the model and the data trend. One can always circumvent the problem by applying the same filtering technique to the model variables and the data series. For example, the researcher can generate $\widehat{m}_t$, apply HP filter to the generated $\widehat{m}_t$ and match the moments of filtered $\widehat{m}_t$ to the moments of HP filtered data. Although this procedure does not have Problem (DD), it is much slower and less efficient than alternative methods we describe below.

Second, Problem (MTS) arises only when the assumption about the trend in the model is different from the actual trend in the data, and this assumption has to be made before estimation. A solution to Problem (MTS) is a flexible framework that nests DT and ST so that the researcher does not have to take a stand on whether $\rho_z < 1$ or $\rho_z = 1$. These two observations suggest that to address Problems (DD) and (MTS), the researcher needs an approach that *i*) transforms the data and model variables in the same way and *ii*) yields stationary series for all $|\rho_z| \leq 1$.

3 Three Robust Approaches

In this section, we consider three approaches that are robust to whether shocks are permanent or transitory. The key to robustness is to filter both the model variables and the observed data consistently so that filtered series are stationary and have the same properties.

3.1 The Quasi-Differenced DT Model

The first method solves an alternative representation of the same model. To begin, recall that the DT model solves the following system of equations:

$$E_t \Gamma_2^D \widehat{m}_{t+1} = \Gamma_0^D \widehat{m}_t + \Gamma_1^D \widehat{m}_{t-1} + \Psi_2^D u_{t+1}^z + \Psi_0^D u_t^z$$

where the Γ and Ψ matrices are defined in (1). Let $\Delta^{\rho_z} = 1 - \rho_z L$ be the quasi-differencing operator. Then for a given ρ_z, a quasi-differenced representation of the DT model can be obtained by multiplying both sides of each equation in (1) by Δ^{ρ_z}:

$$
E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta^\rho \widehat{c}_{t+1} \\ \Delta^\rho \widehat{k}_{t+1} \\ \Delta^\rho \widehat{l}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha - 1 \end{bmatrix} \begin{bmatrix} \Delta^\rho \widehat{c}_{t} \\ \Delta^\rho \widehat{k}_{t} \\ \Delta^\rho \widehat{l}_{t} \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \Delta^\rho \widehat{c}_{t-1} \\ \Delta^\rho \widehat{k}_{t-1} \\ \Delta^\rho \widehat{l}_{t-1} \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ -A_0 \\ 0 \end{bmatrix} \Delta^\rho u^z_{t+1} + \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha - 1 \end{bmatrix} \Delta^\rho u^z_t. \tag{7}
$$

Since $u^z_t = \rho_z u^z_{t-1} + e^z_t$, we have

$$
E_t \Gamma^D_2 \Delta^{\rho_z} \widehat{m}_{t+1} = \Gamma^D_0 \Delta^{\rho_z} \widehat{m}_t + \Gamma^D_1 \Delta^{\rho_z} \widehat{m}_{t-1} + \Psi^D_1 e^z_{t+1} + \Psi^D_0 e^z_t
$$

where $\Delta^{\rho_z} \widehat{m}_t = (\Delta^{\rho_z} c_t, \Delta^{\rho_z} k_t, \Delta^{\rho_z} l_t)$. Note that the error term in (7) is an i.i.d. innovation and therefore $\Delta^{\rho_z} \widehat{m}_t$ is stationary for all $|\rho_z| \leq 1$. The appeal of the quasi-differenced representation is that it is valid for all ρ_z less than or equal to one; (7) is just a special case of (2) at $\rho_z = 1$. Partition $\Theta = (\Theta^-, \rho_z)$. The deep parameters can be estimated as follows:

The QD Estimator: Initialize ρ_z.

1: Quasi-difference the observed data with ρ_z to obtain

$$
c^c_t = \Delta^{\rho_z}(c_t - \bar{g}t), \quad k^c_t = \Delta^{\rho_z}(k_t - \bar{g}t), \quad l^c_t = \Delta^{\rho_z} l_t, \tag{8}
$$

and let $\Delta^{\rho_z} d^c_t = (c^c_t, k^c_t, l^c_t)$.[9]

2: Compute $\widehat{\Omega}^d_{\Delta^{\rho_z}}(0) = cov(\Delta^{\rho_z} d^c_t)$, the covariance matrix of $\Delta^{\rho_z} d^c_t$, and the autocovariance matrix $\widehat{\Omega}^d_{\Delta^{\rho_z}}(1)$. Define $\widehat{\omega}^d_{\Delta^{\rho_z}} = (vech(\Omega^d_{\Delta^{\rho_z}}(0))' \;\; vec(\Omega^d_{\Delta^{\rho_z}}(1))')'$;

3: For a given ρ_z and Θ^-, solve (7) to yield

$$
\Delta^\rho \widehat{m}_t = \Pi_{\Delta^\rho} \Delta^\rho \widehat{m}_{t-1} + B_{\Delta^\rho} \Delta^\rho u^s_t.
$$

Using this equation, compute $\Omega^m_{\Delta^{\rho_z}}(0)$ and $\Omega^m_{\Delta^{\rho_z}}(1)$, the model implied covariance and autocovariance matrices. Define $\widehat{\omega}^m_{\Delta^{\rho_z}} = (vech(\Omega^m_{\Delta^{\rho_z}}(0))' \;\; vec(\Omega^m_{\Delta^{\rho_z}}(1))')'$;

4: Find the structural parameters $\widehat{\Theta} = \arg\min_\Theta \left\| \widehat{\omega}^d_{\Delta^{\rho_z}} - \omega^m_{\Delta^{\rho_z}}(\Theta) \right\|$.

[9]Since projecting series on linear trend yields super-consistent estimates of the coefficient on the time trend, one can ignore the error induced by removing the linear time trend when he or she applies standard asymptotic inference. Likewise, one can introduce structural breaks in a trend directly at this step.

Note that ρ_z and Θ^- are estimated simultaneously. The quasi-differenced estimator differs from the covariance estimator of the previous section in one important respect. The parameter ρ_z now affects both the moments of the model and the data since the latter are computed for the quasi-transformed data. Conceptually, the crucial feature is that the quasi-transformed data are stationary irrespective of ρ_z. Thus, the QD estimator resolves Problem (DD) by applying the same transformation (filter) to the data and model and tackles Problem (MTS) by using a transformation that yields stationary series for any $|\rho_z| \leq 1$. Because $\Delta^{\rho_z} m_t$ is stationary, the estimation problem can be studied under the assumptions of extremum estimation. Under regularity conditions, standard $\sqrt{T}$ asymptotic normality results hold (see Section 3.4).

At this point it is useful to relate our approach with other methods considered in the literature. Fukac and Pagan (2006) propose using Beveridge-Nelson decomposition to estimate and remove permanent component in the data series. Apart from the fact that the permanent component in the Beveridge-Nelson decomposition may be different from actual trend and is subject to stringent assumptions, the clear advantage of our approach is that it is a one-step procedure that can handle multiple I(1) shocks.

Our method is similar to Cogley's (2001) in that neither requires the researcher to take a stand on the properties of the trend dynamics before estimation, but there are important differences. First, quasi-differencing can easily handle multiple I(1) or highly persistent shocks. In contrast, using cointegration relationships works only for certain types of shocks. For example, if the shock to disutility of labor supply is an I(1) process, there is no cointegration vector to nullify a trend in hours. Second, cointegration often involves estimating identities and therefore the researcher has to add an error term (typically measurement error) to avoid singularity. Our approach does not estimate specific equations and hence does not need to augment the model with additional, atheoretical shocks. Finally, using unconditional cointegration vectors may make estimation of some structural parameters impossible. For instance, the parameters governing short-run dynamics such as adjustment costs may be not estimated in this setting because the term due to adjustment costs is zero on average by construction (i.e., adjustment cost is typically zero in steady state). In contrast, our approach utilizes short-run dynamics in estimation and thus can estimate the parameters affecting short-run dynamics of the variables. Overall, our approach can be used in a broader array of situations and we exploit different properties of the model in estimation.

3.2 The ΔDT Model

Naturally, if elements of $\Delta^{\rho_z} \widehat{m}_t$ are stationary concepts when $|\rho_z| \leq 1$, they are also stationary when the data are quasi-differenced at $\rho_z = 1$. This suggests that solving the first difference representation of the DT model will also yield robust estimates. The corresponding system of equations is

$$E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\widehat{c}_{t+1} \\ \Delta\widehat{k}_{t+1} \\ \Delta\widehat{l}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha-1 \end{bmatrix} \begin{bmatrix} \Delta\widehat{c}_{t} \\ \Delta\widehat{k}_{t} \\ \Delta\widehat{l}_{t} \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \Delta\widehat{c}_{t-1} \\ \Delta\widehat{k}_{t-1} \\ \Delta\widehat{l}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ -A_0 \\ 0 \end{bmatrix} \Delta u^z_{t+1} + \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha-1 \end{bmatrix} \Delta u^z_t. \tag{9}$$

Observe that when $\Delta u_t^z = (\rho_z - 1)u^z_{t-1} + e^z_t$, and $\rho_z < 1$, ρ^z remains a parameter of the model (3) unless it is constrained to be one. To stress that ρ_z is a free parameter and contrast it with the constrained specification, we do not put a superscript on the first difference operator. The difference between the constrained $\Delta^1 DT$ and unconstrained ΔDT models is that the unconstrained model is valid whether or not $\rho_z = 1$, while the constrained model is an alternative representation of the ST model and is thus valid only when $\rho_z = 1$. Note that the QD estimator and ΔDT estimator are equivalent when $\rho_z = 1$. While the moments of ΔDT model are robust to whether ρ_z is on the unit circle, this approach is less efficient relative to quasi-differencing since the data will be over-differenced when the data are already stationary. The estimation procedure for the unconstrained ΔDT estimator is as follows:

The ΔDT Estimator:

1: First difference the observed data to obtain

$$c^c_t = \Delta c_t - \bar{g}, \quad k^c_t = \Delta k_t - \bar{g}, \quad l^c_t = \Delta l_t. \tag{10}$$

and let $\Delta d^c_t = (c^c_t, k^c_t, l^c_t)$.

2: Compute $\widehat{\Omega}^d_\Delta(0) = cov(\Delta d^c_t)$, the covariance matrix of Δd^c_t, and the autocovariance matrix $\widehat{\Omega}^d_\Delta(1)$. Define $\widehat{\omega}^d_\Delta = (vech(\Omega^d_\Delta(0))' \;\; vec(\Omega^d_\Delta(1))')'$;

3: For a given Θ, solve (9) and compute $\Omega^m_\Delta(0)$ and $\widetilde{\Omega}^m_\Delta(1)$, the model implied covariance and autocovariance matrices. Define $\widehat{\omega}^m_\Delta = (vech(\Omega^m_\Delta(0))' \;\; vec(\Omega^m_\Delta(1))')'$;

4: Find the structural parameters $\widehat{\Theta} = \arg\min_\Theta \left\| \widehat{\omega}^d_\Delta - \omega^m_\Delta(\Theta) \right\|$.

3.3 The HP-HP Model

The final robust method is based on the HP filter. A desirable feature of the HP filter is that it can remove deterministic as well as stochastic trends. As discussed in King and Rebelo (1993), the data can be rendered stationary without the user deciding a priori the specific type of non-stationarity that is to be handled. The HP filter is heavily used in empirical analysis, but as seen earlier, the common practice of estimating either the DT or the ST model using HP filtered the data can lead to substantial bias in the parameter estimates. The reason is that the HP filter changes the autocovariance structure of the data. It follows that if we were to filter the data, we would also need to simultaneously HP filter the model variables.[10]

For the case with stationary variables, define a HP filtered series as $\breve{y}_t = H(L)y_t$, where $H(L)$ is a polynomial in the lag operator. Let $\Omega^m(0)$ and $\Omega^m(1)$ be the variance and first order autocovariance of the untransformed model variables. Collect their unique elements of interest into a vector ω^m. Then the autocovariance of the filtered data can be computed as

$$\breve{\omega}^m_{HP,s}(k) = \sum_{i=-\infty}^{\infty} \sum_{j=\infty}^{\infty} H_i H_j \omega^m_s(k+i-j).$$

We can compute only $H_j, j = 1, \dots T$. Furthermore, the H_j computed from T observations are different at the beginning/end of the sample and the middle of the sample. This problem, also discussed in Christiano and den Haan (1996), has no perfect solution. One possibility is to use the weights in the middle of the sample throughout. An alternative is to simulate the theoretical moments as implied by the model. Both will yield approximation to the analytical covariances.

If y_t is not stationary because of deterministic or stochastic trends, which is the case of interest, one encounters the additional problem that the population autocovariances of model variables are not well defined, even though sample autocovariances for non-stationary processes can be mechanically calculated. Since analytic calculation of the moments of HP filtered series is burdensome even in simple univariate cases, we use simulations to approximate the moments of the HP filtered model and to estimate structural parameters as follows:

The HP-HP Estimator: Initialize ρ_z.

1: Let $\breve{d}^c_t$ be the HP filtered data. Compute $\Omega^d_{HP}(0)$ and $\Omega^d_{HP}(1)$, the variance and autocovariance matrix of $\breve{d}^c_t$. Define $\omega^d_{HP} = (vech(\Omega^d_{HP}(0))' \;\; vec(\Omega^d_{HP}(1))')'$;

2: For a given guess of Θ, solve the DT model for $\widehat{m}_t$, let $m_t = \widehat{m}_t + \bar{g}t$, draw shocks $\{e^z_t\}_{t=1}^S$ and simulate data $\{m_t\}_{t=1}^S$, where S is sufficiently large to make sampling error in Step 3 small;

3: Apply HP filter to $\{m_t\}_{t=1}^S$ and get cyclical component $\breve{m}_t$. Compute $\Omega^m_{HP}(0)$ and $\Omega^m_{HP}(1)$, the variance and autocovariance matrix of $\breve{m}_t$, and define $\omega^m_{HP} = (vech(\Omega^m_{HP}(0))' \;\; vec(\Omega^m_{HP}(1))')'$;

[10] Although we focus on HP filter, alternative filters that remove I(1) component in the series can be used in estimation as long as the same filter is applied to model and data series. One may use band-pass filter described in, for example, **?**.

4: Find the structural parameters $\widehat{\Theta} = \arg\min_{\Theta} \left\| \omega^d_{HP} - \omega^m_{HP}(\Theta) \right\|$.[11]

By construction of the HP filter, both $\breve{d}_t$ and $\breve{m}_t$ are stationary for all $|\rho_z| \leq 1$. Under the DT model, ρ_z is a free parameter. Thus, adding deterministic terms back to the DT model yields a model that can potentially have both deterministic and stochastic trends. At each iteration of Θ, one has to simulate and filter model series, which makes the procedure quite slow relative to other robust approaches (about 3-5 times slower if compared to QD).

The three robust methods can be summarized as follows:

True Model	Assumed Model/ Variables	Data	Estimator
ST, DT	QD, $\Delta^{\rho_z}\widehat{m}_t$	$\Delta^{\rho_z} d^c_t$	QD
ST, DT	ΔDT, $\Delta\widehat{m}_t$	Δd^c_t	ΔDT.
ST, DT	HP, $\breve{m}_t$	$\breve{d}^c_t$	HP-HP

All three methods do not require the researcher to take a stand on whether $\rho_z < 1$ or $\rho_z = 1$ before estimation. The ST and DT are nested within QD, ΔDT and HP-HP framework.

4 Simulations

4.1 Setup and Calibration

We generate the data as either DT (deterministic trends) or ST (stochastic trends) using the model equations for the stationary (i.e., normalized) variables. The model variables are then rescaled back to non-stationary form and treated as observed data $d_t = (c_t, k_t, y_t, l_t)$ that the researcher takes as given. The researcher then decides (i) whether to use the model equations implied by DT or ST for estimation, and (ii) how to detrend the data.

We estimate $\Theta = (\alpha, \rho, \sigma)$ and treat parameters $(\beta, \delta, \theta, \bar{g})$ as known. We calibrate the model as follows: capital intensity $\alpha = 0.33$; disutility of labor $\theta = 1$; discount factor $\beta = 0.99$; depreciation rate $\delta = 0.1$; gross growth rate in technology $\bar{g} = \bar{\gamma} = 1.005$. We restrict the admissible range of the estimates of α to [0.01,0.99]. We vary the persistence of shocks to technology u^z_t. The parameter ρ_z takes values (0.5, 0.95, 0.99, 1). The admissible range for $\widehat{\rho}_z$ in the DT model is [-0.999,0.999]. Since for now we have only one shock in the model, we set the standard deviation of e^z_t to $\sigma = 0.1$ without loss of generality. We perform 1,000 replications for each choice of parameter values. For each replication, we create series with T=300 observations which is a typical sample size in applied macroeconometric analysis.

[11] We experimented with an alternative procedure. For each Θ, we simulated the model to generate $j = 1, ..., N$ samples of size T. For each j we computed moments. Then we averaged moments over j and used this average for ω^m_{HP}. This procedure is much slower and the results are very similar to the procedure we present in the text.

In the simulations, we use a covariance structure estimator that minimizes the distance between the observed unconditional autocovariances of the data and those implied by the model. We described the estimator in the previous sections. To minimize distortions associated with poor estimation of the optimal weighting matrix, we use an identity weighting matrix in our method of moments estimator.[12] In all simulations and for all estimators, we set starting values in optimization routines equal to the true parameter values.

4.2 Results for the Baseline Model

We report simulation results for the baseline one-shock model in Table 2 and present the kernel density estimates for parameter estimates in Figure 1. We use the following notation to label different cases. In case (XX,YY), XX stands for the method used to filter the data, while YY stands for the assumed model. Thus, (LT,$\widehat{m}_t$) means that the data used in estimation are residuals from projection on a time trend, and the assumed model is expressed in terms of $\widehat{m}_t$ with $|\rho_z| < 1$. The DGP is given in the first column.

Our simulations suggest that combinations (QD,$\Delta^{\rho_z}\widehat{m}_t$), (FD,$\Delta\widehat{m}_t$) and $(HP, \check{m}_t)$, which are reported in columns (4), (6) and (7) and correspond to the QD, unconstrained ΔDT, and HP-HP estimators respectively, yield estimates generally centered at the true values. The distribution of the estimates is bell-shaped and well-behaved uniformly for all values of ρ_z. That is, the performance of QD, ΔDT and HP-HP estimators does not change materially as ρ_z approaches one. This pattern is recurrent in all simulations. In contrast, other estimators exhibit significant biases and larger dispersion of estimates especially when ρ_z is close to a unit circle. Below we document their properties and explain why these estimators tend to underperform.

Consider first the (LT,$\widehat{m}_t$) combination when the researcher uses series after linear detrending as the data concept and $\widehat{m}_t$ as the model concept of the observed variables (column (1), Table 2). For small to moderate values of ρ_z, this combination performs well: the distribution of the parameter estimates is centered at true values. However, as ρ_z increases the performance of the (LT,$\widehat{m}_t$) combination quickly deteriorates. There is a significant upward bias in the estimates of the capital share α. Furthermore, this bias increases with ρ_z so much that at $\rho_z = 1$, the mean of $\widehat{\alpha}$ is close to one. The bias in $\widehat{\sigma}$ also worsens rapidly as ρ_z approaches one and the dispersion of the estimates is large as seen from the flat density of $\widehat{\sigma}$ in Figure 1. The estimates of ρ_z tend to be relatively close to true values up to $\rho_z = 0.95$. As ρ_z approaches one, however, there is a strong downward bias in $\widehat{\rho}_z$. For example at $\rho_z = 1$, the mean of $\widehat{\rho}_z$ is approximately 0.7.

Note that the (LT,$\widehat{m}_t$) case can not only significantly bias the estimates but can also yield multi-

[12] Using optimal weighting matrix introduces a small bias in the estimates. The bias vanishes when $T \to \infty$.

modal distribution of the estimates. For example, the case with $\rho_z = 0.99$ has two peaks in the distribution of $\widehat{\alpha}, \widehat{\sigma}$ and $\widehat{\rho}_z$, i.e., the objective function of the covariance structure estimator has two or more local optima. This observation is particularly troubling for users of standard optimization routines as these routines can fail to escape from local optima. Importantly, estimates based on different local optima can lead to drastic changes in the economic interpretation of the estimates.

The case of $\rho_z = 1$ (last row of the (LT,$\widehat{m}_t$) column) is particularly interesting because linear detrending is commonly used in estimation of DSGE models, as seen from Table 1. Projecting a series with a unit root on time trend is known to lead to spurious cycles in univariate analysis (e.g. Nelson and Kang (1981)). Our results suggest that in systems estimation such as the one considered here, linear detrending leads to extremely strong biases in the estimates of the structural parameters. Since technology shocks appear to be highly persistent and well approximated with unit root (Problem MTS), researchers should be very cautious with using linearly detrended data for estimation of DSGE models in applied work.

Turning to the (HP,$\widehat{m}_t$) combination in column (2), the estimates of ρ_z have a strong downward bias. On the other hand, there is a strong upward bias in $\widehat{\alpha}$ and $\widehat{\sigma}$.[13] These estimates suggest larger but less persistent shocks to technology as well as a significant role of capital as a mechanism for propagating shocks in the model. To understand this pattern, recall that HP filter removes not only the linear trend but also low frequency variation in the series. When ρ_z is large, HP filter can significantly alter the properties of the series. More specifically, HP filter changes not only the persistence of the series (recall Cogley and Nason (1995)) but also the relative volatility and serial correlation of the series (see e.g. King and Rebelo (1993) and Harvey and Jaeger (1993)). This translates into biased estimates of all parameters because the estimator is forced to match the properties of the altered data which is different from the model concept of observed variables.

Under (HP,$\widetilde{m}_t$), ρ_z is fixed at 1 and the model variables are $\widetilde{m}_t$. As seen from column (3), the estimates of α and σ remain unsatisfactory. The estimates of α's are lumped at the boundary of the admissible range [0.01, 0.99] and the estimates of σ tend to be very close to zero. In other words, the estimated model suggests that shocks to technology are very small but the propagation through capital accumulation is strong. Why does this happen? Note that in the ST model defined in terms of $\widetilde{m}_t$, the dynamics of the variables tend to have weak serial correlation because deviations from u_t are transitory and dissipate quickly as variables such as consumption adjust to almost full strength in response to permanent shocks to technology. On the other hand, HP filter leaves out sizable serial correlation in the filtered data. Thus, the fitted model is forced to produce parameter values that have

[13] Note that we do not HP-filter labor series as labor is stationary irrespective of whether $\rho_z < 1$ or $\rho_z = 1$. Results do not change qualitatively when we estimate the model using HP-filter labor series.

strong propagation mechanism to generate relatively strong serial correlation in deviations from the stochastic trend.

To understand the strong downward bias in $\widehat{\sigma}$, note that $var(\widetilde{y}_t) < var(\widetilde{c}_t)$ and $var(\widetilde{k}_t) > var(\widetilde{c}_t)$ in the model, while $var(y^c_t) < var(c^c_t)$ and $var(k^c_t) < var(c^c_t)$ in the HP-filter series. As α approaches one, $var(\widetilde{y}_t)$ and $var(\widetilde{k}_t)$ become approximately equal to $var(\widetilde{c}_t)$ and thus the gap in the relative volatility between output, capital and consumption resembles the relative volatility in the data. At the same time, larger values of α increase the volatility of the series and the estimator decreases the size of the shocks to match the level of volatility in the data. Hence, there is a strong downward bias in $\widehat{\sigma}$. Overall, results for combinations (HP,$\widehat{m}_t$) and (HP,$\widetilde{m}_t$) suggest that Problems (DD) and (MTS) can significantly affect estimates of structural parameters and can lead to erroneous economic interpretations.

To get a sense of how much difference filtering can make, consider the combination (FD,$\Delta^1 m_t$), reported in column 5, Table 2. It performs reasonably well when $\rho_z \approx 1$, that is, ST is the correct assumption and first differencing is correctly applied to data and model variables. As ρ_z departs from one, Problem (MTS) manifests in an increasing upward bias in both $\widehat{\alpha}$ and especially $\widehat{\sigma}$. Note that despite the fact that the estimates based on (FD,$\Delta^1 m_t$) exhibit sizable biases when ρ_z moves away from one, (FD,$\Delta^1 m_t$) dominates (HP,$\widetilde{m}_t$) by a large margin. This pattern is typical in our simulations.

4.3 Estimation of the Propagation Mechanisms

Clearly, the absurdly large estimates of α or similar problems with the estimates of deep parameters can alert the researcher that the model is likely misspecified and he or she must make adjustments to the model. One possible and popular modification is to introduce serial correlation in the growth rates of structural shocks such as technology. Interestingly, when we introduce such correlation in the growth rates of technology (not reported) and estimate the model using (HP,$\widetilde{m}_t$) combination, the estimates of α take more plausible values in the range of 0.4-0.5. However, this modification in the model is ad hoc and more importantly it indicates that improper choice of filtering techniques and model concepts can induce the researcher to augment correctly specified models with spurious mechanisms of propagation and amplification to match the moments of the data.[14]

To highlight this point, we augment the basic model with internal habit in consumption. This modification is a popular way to introduce greater persistence and amplification in business cycle models. Specifically, consider an alternative utility function:

$$\max \sum \beta^t \Big[\ln(C_t - \phi C_{t-1}) - L_t \Big]$$

[14]For example, Doorn (2006) shows in simulations that HP filtering can significantly alter the parameter estimates governing dynamic properties in his inventory model.

where $\phi \in [-0.999, 0.999]$ measures the degree of habit in consumption. In this model, the researcher estimates $(\alpha, \phi, \rho, \sigma)$. We set $\phi = 0.8$ to investigate how the treatment of the trends affect estimates of internal propagation mechanisms as well as estimates of other structural parameters. We report results in Table 3 and Figure 2. To save space, we do not consider the case of $\rho_z = 0.5$ and present kernel densities of the estimates for only the case of $\rho_z = 0.95$ in Figure 2.

Similar to the results in the previous section, QD, ΔDT and HP-HP perform well. The bias in the estimates is generally negligible and the distribution of the estimates is well-behaved. Overall, QD, ΔDT and HP-HP strongly dominate alternative estimators whose performance we examine below.

The combination (LT,$\widehat{m}_t$) has a relatively small upward bias in $\widehat{\phi}$ when $\rho_z = 0.95$ but the performance of (LT,$\widehat{m}_t$) quickly deteriorates as ρ_z approaches one, see column (1) of Table 3. Specifically, at $\rho_z = 0.99$ the mean value of $\widehat{\phi}$ is close to the the true value of ϕ but the dispersion of $\widehat{\phi}$ rapidly increases indicating that the distribution of $\widehat{\phi}$ is quite flat. When $\rho_z = 1$, the mean of $\widehat{\phi}$ sharply drops and the dispersion of $\widehat{\phi}$ increases further. In fact, the kernel density of $\widehat{\phi}$ is practically flat (not reported) so that researcher using (LT,$\widehat{m}_t$) may end up with effectively any estimate of ϕ. Note that introducing habit formation changes the pattern of biases in the estimates of other parameters when compared to the baseline model without habit formation. In particular, although $\widehat{\alpha}$ is upwardly biased in the model with and without habit formation, there is a downward bias in $\widehat{\rho}_z$ and $\widehat{\sigma}$ for the model with habit formation which is different from the results for the baseline model without habit formation. Note that it is very hard to predict the sign of the bias in general. Small modifications in a model can lead to distortionary effects reinforcing or attenuating each other so that estimates can over- or understate the magnitudes of structural parameters. The direction of the bias is highly model specific and a priori ambiguous.

Under the (HP,$\widehat{m}_t$) combination, when the researcher uses HP filter to remove the trend, there are larger distortions to $\widehat{\phi}$. The estimate of ϕ has a clear upward bias when $\rho_z = 0.95$. However, the mean value of $\widehat{\phi}$ understates the degree of the bias as the distribution of $\widehat{\phi}$ has a thick left tail. As ρ_z approaches one, the dispersion of $\widehat{\phi}$ increases dramatically, which is a manifestation of the flat distribution of $\widehat{\phi}$. This finding suggests that identification of ϕ from the filtered data may be poor. Indeed, identification of ϕ comes from low frequency variation in the data but this frequency is removed or greatly attenuated by the HP filter. Other estimates are also biased. In particular, $\widehat{\alpha}$ has a stronger upward bias than in the case without habit formation. The bias in $\widehat{\sigma}$ decreases with ρ_z while it increases with ρ_z in the model without habit formation. Also note that the extent of the bias in $\widehat{\rho}_z$ is smaller in this model than in the model without habit formation.

The (HP,$\widetilde{m}_t$) combination which imposes $\rho_z = 1$ tends to produce results similar to the previous case but with more acute identification problems for ϕ as the density of $\widehat{\phi}$ is fairly flat even for

$\rho_z = 0.95$. In addition, the mean of $\widehat{\alpha}$ is away from the boundary of the admissible space for estimates of α because a part of the upward bias in $\widehat{\alpha}$ is absorbed by changes in the estimates of ϕ. Interestingly, as ϕ increases towards one, the bias in $\widehat{\alpha}$ turns from upward to downward. Again, note that using an alternative combination (FD,$\Delta^1 m_t$) can greatly improve the estimates when $\rho_z \approx 1$, which is similar to the baseline case without habit formation.

4.4 Statistical Properties

The above simulations indicate that our quasi-differenced estimates are close to the true value and the difference between the estimates and the true value is by and large symmetrically distributed. Note that our quasi-differenced estimator is nothing but a non-linear GMM estimator using an identity as weighting matrix and stationary (after quasi-differencing) data. One may wonder whether the t-statistic for estimates based (QD,$\Delta^{\rho_z}\widehat{m}_t$) (or (FD,$\Delta\widehat{m}_t$), or (HP,$\breve{m}_t$)) are well approximated by the normal distribution for large T and our estimator is $\sqrt{T}$ consistent and asymptotically normal. In this subsection, we assess this conjecture.

Let $v_t^c = (d_t^c, d_{t-1}^c, \ldots, d_{t-s}^c)$ be the vector of stacked data so that $\omega^d = vech(cov(v_t^c))$. Likewise, define v_t^m, the vector of stacked model variables, and $\omega^m = vech(cov(v_t^m))$. Define

$$\frac{1}{T}\sum_{t=1}^{T} g_t = \bar{g} = \omega^d - \omega^m.$$

By assumption, g_t is continuous in Θ, which is compact and $g_t(\Theta_0)$ is stationary ergodic when evaluated at the true parameter vector, $\Theta_0 = (\Theta_0^-, \rho_0)$. Let G be the matrix of derivatives of g with respect to Θ. Then if $\widehat{\Theta} = \text{argmin}_\Theta\, J = \bar{g}'\bar{g}$,

$$\sqrt{T}(\widehat{\Theta} - \Theta_0) \xrightarrow{d} A(\rho_0) N(0, S)$$

where $A(\rho_0) = (G'G)^{-1}G'$ is non-stochastic when the data are stationary and has a random limit when $\rho_0 = 1$. Although $\widehat{\Theta}$ is mixed normal, the t statistic is asymptotically normal. In large samples, the simulated density for t-statistic should be close to the p.d.f. of the standard normal random variable. We focus on four combinations (QD,$\Delta^{\rho_z}\widehat{m}_t$), (FD,$\Delta\widehat{m}_t$), (HP,$\breve{m}_t$) and (LT,$\widehat{m}_t$) and report the kernel density of t-statistic for T=300 and T=2,000 in Figures 3 and 4. We employ Newey-West estimator of S and compute t-statistic for the parameters of the baseline one-shock model.

The figures show that the distribution based on the QD estimator for α and σ is generally close to the N(0,1) density. Likewise, apart from the case when $\rho_z = 1$, the distribution of the t-statistic for $\widehat{\rho}_z$ is also closely approximated by the standard normal distribution.[15] The distribution of t-statistic

[15] Note that the model does not have a unique rational expectations equilibrium if $\rho_z > 1$. Since we consider only

based on ΔDT is somewhat less impressive but nonetheless it is a much better approximation to N(0,1) than the approximation provided by the more commonly used (LT,$\widehat{m}_t$) combination. HP-HP and QD have similar performance.

While normality of $\widehat{\alpha}$ and $\widehat{\sigma}$ may not seem surprising, approximate normality of $\widehat{\rho}_z$ when ρ_z is close to one may be unexpected. This is because the literature on integrated regressors suggest a convergence rate of T, but that the asymptotic distribution is highly skewed. This surprising and curious result was investigated in more detail in Gorodnichenko and Ng (2007) in simpler linear regression models. In a nutshell, our sample moments are stationary when evaluated at the true parameter vector to permit application of central limit theory, ie. $\sqrt{T}\bar{g}(\Theta_0)\xrightarrow{d}N(0,S)$. The effect of non-stationarity is then confined to the Jacobian matrix with the consequence that $\widehat{\Theta}-\Theta_0 \approx A(\rho_0)N(0,S)$, where $A(\rho_0)$ is non-random if the data are stationary but is a function of Brownian motions if the data are strongly persistent and possibly non-stationary. However, the t statistics are approximately normal, which greatly facilitates inference. The cost, which seems warranted, is that the estimator looses its super-consistent property, though it continues to be $\sqrt{T}$ consistent.

5 The General Formulation

Our Monte Carlo experiments suggest that the (QD,$\Delta^{\rho_z}\widehat{m}_t$), (FD,$\Delta\widehat{m}_t$) and (HP,$\breve{m}_t$) estimators outperform popular alternatives in terms of providing less dispersed and biased estimates of structural parameters. Combinations (QD,$\Delta^{\rho_z}\widehat{m}_t$), (FD,$\Delta\widehat{m}_t$) and (HP,$\breve{m}_t$) are similar in terms of having little or no bias. However, (QD,$\Delta^{\rho_z}\widehat{m}_t$) and (HP,$\breve{m}_t$) tend to have smaller dispersion and better behaved distribution of the estimates than (FD,$\Delta\widehat{m}_t$) when at least one of the shocks is persistent.

The key to robustness is that these estimators do not depend on whether the data are persistent and that they apply basic transformations to both possibly persistent data and model variables so that there is a coherent mapping between the model and data. Effectively, Problems (DD) and (MTS) are addressed simultaneously because *i*) the researcher does not have to take a stand on the properties of the forcing variables (e.g., technology has a permanent or transitory shocks) as the transformed data and model variables are stationary for all parameter values describing persistence of forcing variables; and *ii*) the researcher applies the same transformation (filter) to the model and the data to make the variables stationary so that filtered variables in the model and data have the same connotation. In other words, (QD,$\Delta^{\rho_z}\widehat{m}_t$), (FD,$\Delta\widehat{m}_t$), and (HP,$\breve{m}_t$) preserve consistency between the data and model concepts irrespective of whether variables are stationary or not.

parameter values consistent with the existence of a unique rational expectations equilibrium, the estimate for ρ_z becomes distorted as $\widehat{\rho}_z$ cannot exceed unity. However, in the linear single-equation case considered in Gorodnichenko and Ng (2007), the distribution of $\widehat{\rho}_z$ is well behaved and close to the standard normal since $\widehat{\rho}_z$ is allowed to exceed one.

Our QD framework straightforwardly extends to more general cases. Suppose there are J shock processes u_{jt}, $j = 1, \ldots J$, and

$$(1 - \rho_j L) u_{jt} = e_{jt}, \quad j = 1, \ldots J$$

and some of the ρ_j may be on the unit circle. Let m_t be a vector of (predetermined, non-predetermined, plus exogenous) variables of the model and let $\widehat{m}_t$ be the vector of zero-mean variables that are deviations of m_t from the steady state values. Normalization like $\widetilde{m}_t = \widehat{m}_t - u_t$ or transformation like $\Delta \widehat{m}_t$ is necessary to make model variables stationary when some shocks are permanent. As seen from Table 1, most studies assume that shocks are transitory and solve

$$\Gamma_0^D \widehat{m}_t = \Gamma_1^D E_t \widehat{m}_{t+1} + \Gamma_2^D \widehat{m}_{t-1} + \Psi_1^D u_{t+1} + \Psi_0^D u_t$$

A smaller number of studies assume stochastic trends and solve

$$\Gamma_0^S \Delta \widehat{m}_t = \Gamma_1^S E_t \Delta \widehat{m}_{t+1} + \Gamma_2^S \Delta \widehat{m}_{t-1} + \Psi_1^S e_{t+1} + \Psi_0^S e_t.$$

As illustrated in Section 2, the estimation approach depends on what is the assumed model, and how the observed data are filtered to become stationary. Let d_t be a vector of r observed variables. The general solution in state space representation is

$$d_t = \delta_0 + \delta_1 t + B \, \widehat{m}_t \tag{11}$$

The measurement equation (**??**) links d_t to the q model variables $\widehat{m}_t$ through the matrix B $(r \times q)$.[16] The parameters δ_0 and δ_1 are $r \times 1$ vectors of restricted constants to be estimated along with the other parameters. This ensures that the data are detrended using model consistent parameters. An alternative, used in Ireland (2004a) and many others, is to linearly detrend the data prior to estimation. This amounts to not imposing constraints on δ_0 and δ_1 to take values implied by the model. If the model is correctly specified for the data, both methods of detrending are asymptotically equivalent. Without loss of generality, simply let $d_t^c = d_t - \delta_0 - \delta_1 t$ be the detrended data so that the measurement equation becomes

$$d_t^c = d_t - \delta_0 - \delta_1 t = B \, \widehat{m}_t. \tag{12}$$

There are alternative measurement equations. For example, the measurement equation for first-differenced data is $d_t^c = \Delta d_t - \delta_1 = B \, \Delta \widehat{m}_t$. Likewise, the measurement equation for HP-HP estimator simply applies HP filter to both sides of equation (**??**).

[16] A vector of r measurement errors η_t can be added to the measurement equation as in Edge et al. (2005).

The quasi-differencing approach can be extended to the case of multiple shocks by defining

$$\Delta^{\rho}(L) = \prod_{j=1}^{J}(1 - \rho_j L).$$

Now the quasi-differencing operator is the product of the J polynomials in lag operator that describes the dynamics of the J shocks. Then the J-shock quasi-differenced model is defined as

$$E_t \Gamma_2^D \Delta^{\rho} \widehat{m}_{t+1} = \Gamma_0^D \Delta^{\rho} \widehat{m}_t + \Gamma_1^D \Delta^{\rho} \widehat{m}_{t-1} + \Psi_1^D \Delta^{\rho} u_{t+1} + \Psi_0^D \Delta^{\rho} u_t, \tag{13}$$

The link between the data and the model is given by

$$\Delta^{\rho}(d_t - \delta_0 - \delta_1 t) \quad = \quad B \, \Delta^{\rho} \widehat{m}_t.$$

This link is valid whether the true model is DT or ST. Note that one does not have to work with the product of $(1 - \rho L)$ operators for each shock. Following the insight from the comparison of the QD and unconstrained ΔDT estimators, one can use quasi-differencing only for shocks that are expected to be persistent and do not transform the data further to accommodate other, known-to-be stationary shocks. For example, if one knows that shocks to tastes dissipate quickly while technology shocks z_t are highly persistent, the researcher can use only $(1 - \rho_z L)$ in the Δ^{ρ} operator.

5.1 A Two Shock Example

To illustrate the multiple shock case, we re-introduce shocks to hours in the model so that the system is given by

$$\begin{aligned} E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \\ \widehat{l}_{t+1} \end{bmatrix} \quad &= \quad \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha - 1 \end{bmatrix} \begin{bmatrix} \widehat{c}_t \\ \widehat{k}_t \\ \widehat{l}_t \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \widehat{c}_{t-1} \\ \widehat{k}_{t-1} \\ \widehat{l}_{t-1} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ -A_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{t+1}^z \\ u_{t+1}^q \end{bmatrix} + \begin{bmatrix} 1 - \alpha & 1 \\ 0 & 0 \\ \alpha - 1 & 0 \end{bmatrix} \begin{bmatrix} u_t^z \\ u_t^q \end{bmatrix} \end{aligned} \tag{14}$$

Following the procedures we describe above, it is straightforward to write this model in terms of stationary variables $\widetilde{m}_t$ or Δm_t when shocks to technology or hours contain a unit root.

Let $\Delta^\rho = (1-\rho^z L)(1-\rho^q L)$. The quasi-differenced representation of the two-shock model (15) is:

$$E_t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & A_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta^\rho \widehat{c}_{t+1} \\ \Delta^\rho \widehat{k}_{t+1} \\ \Delta^\rho \widehat{l}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\alpha \\ 1 & A_0 & 0 \\ A_1 & A_2 & \alpha-1 \end{bmatrix} \begin{bmatrix} \Delta^\rho \widehat{c}_{t} \\ \Delta^\rho \widehat{k}_{t} \\ \Delta^\rho \widehat{l}_{t} \end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & A_4 & 0 \end{bmatrix} \begin{bmatrix} \Delta^\rho \widehat{c}_{t-1} \\ \Delta^\rho \widehat{k}_{t-1} \\ \Delta^\rho \widehat{l}_{t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ -A_0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^z_{t+1} \\ e^q_{t+1} \end{bmatrix} + \begin{bmatrix} 1-\alpha & 1 \\ \rho^q A_0 & 0 \\ \alpha-1 & 0 \end{bmatrix} \begin{bmatrix} e^z_{t} \\ e^z_{t} \end{bmatrix}$$
$$+ \begin{bmatrix} -\rho^q(1-\alpha) & -\rho^z \\ 0 & 0 \\ \rho^q(\alpha-1) & 0 \end{bmatrix} \begin{bmatrix} e^z_{t-1} \\ e^q_{t-1} \end{bmatrix}.$$

This representation is valid when none, one or both shocks are non-stationary. It is easy to see that when none or one shock is permanent, the transformation recovers the correct representation. When both shocks are permanent, the representation is the first difference of (15). Instead of solving for a model that is possibly non-stationary, we solve for a model that is possibly over-differenced.

In this model, we estimate $(\alpha, \rho_z, \sigma_z, \rho_q, \sigma_q)$. The relative persistence and variability of shocks is important for the estimates. We fix $\sigma_z = 0.1$ and let σ_q take values (0.025,0.05,0.15). The persistence of the shocks to technology and hours is described by the vectors (0.95,0.99,1) and (0.5,0.8,0.9,0.975) respectively so that in our exercise technology shocks are generally more persistent than shocks to hours. To preserve space, we report only selected results in Table 4 and Figures 5 through 7 ($\rho_q =$ 0.8,$\rho_z = 0.95$) and provide only a concise summary of the results. Additional results are available upon request.

In short, combinations (QD,$\Delta^{\rho_z}\widehat{m}_t$), (FD,$\Delta\widehat{m}_t$), and (HP,$\breve{m}_t$) perform well while other estimators have significantly worse performance. Using HP filter to remove the trend as in (HP,$\widehat{m}_t$) or (HP,$\widetilde{m}_t$) continues to induce very strong biases in all estimates because the notion of trend is different in the model and in the data. The combination (FD,$\Delta^1 m_t$) performs well when technology shocks have a unit root but its performance quickly deteriorates as ρ_z moves away from one. Linear detrending in (LT,$\widehat{m}_t$) can perform relatively well when shocks to stationary hours are large relative to technology shocks. That is, as shocks to hours explain a larger fraction of variation in variables, identification of structural parameters improves as one can rely on variation in the stationary, non-persistent structural shocks. For example, in the case of $\rho_z = 0.95$, as σ_q increases from 0.025 to 0.15 the mean estimate of α falls from 0.4520 to 0.3813 so that the bias decreases from 0.1220 to 0.0513. The reduction in the bias is even more dramatic when ρ_z is closer to one. The bias in other estimates exhibits a similar pattern. However, the biases become pronounced again when shocks to hours become more persistent, i.e., ρ_q increases towards one. The pair (FD,$\Delta^1 m_t$) dominates (HP,$\widehat{m}_t$) or (HP,$\widetilde{m}_t$), either of which continues to exhibit strong biases in the estimates. In some cases (e.g., (HP,$\widetilde{m}_t$)) the relative size of the shocks is reversed in the estimates—that is, $\widehat{\sigma}_q > \widehat{\sigma}_z$ while $\sigma_q < \sigma_z$—so that the researcher may be tempted to conclude that shocks to hours have larger volatility than shocks to technology while the opposite is true.

6 Concluding Remarks

This paper has several substantive findings. First, the paper identifies Problems (DD) and (MTS) and shows that the consequences of these two problems can be devastating for the estimates of structural parameters in DSGE models. Specifically, the paper demonstrates that Problems (DD) and (MTS) can lead to distorted estimates, spurious estimates of propagation/amplification mechanisms (both external and internal), poor identification of structural parameters (especially parameters identified from low frequency variation). Importantly, both Problem (DD) and Problem (MTS) are empirically relevant and often arise in applied work.

Second, the paper proposes a robust approach to address Problems (DD) and (MTS) simultaneously. The key to robustness is to tackle both problems by applying the same transformation to the data and model variables *and* using the fact that this transformation yields stationary series for all parameters values that can describe persistence of forcing variables in the model. We illustrate the performance of this approach using three transformations: quasi-differencing, first-differencing and Hodrick-Prescott (HP) filter. Other popular filters such as band-pass filter can yield consistent estimates when the researcher applies the same filter to data and model series. At the same time we demonstrate that using HP or other filters to remove the trend in the data and employing different filters for the model variables result in strongly biased estimates of structural parameters. Simulations show that our approach outperforms popular alternatives not only in terms of having smaller bias and smaller dispersion of the estimates but also in terms of providing $\sqrt{T}$ consistent inference even for parameters governing persistence of exogenous shocks. Although the paper illustrates the working of our approach on specific simple examples, the paper also shows that the framework can be easily generalized to more complex settings.

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Table 1. Summary of selected works.

Paper	Technology process	System vs. single equation	Filter	Estimation method
Ireland (2001)	Stationary AR	system	Linear trend	MLE
Del Negro, Schorfheide, Smets, and Wouters (2004)	Unit root with serial correlation in growth rates	system	First difference	Bayesian
Bouakez, Cardia, and J. Ruge-Murcia (2005)	Stationary AR	system	Linear trend	MLE
Faia (2007)	Stationary AR	system	HP	calibration
Clarida, Gali, and Gertler (2000)	Stationary AR	equation	HP and deviation from CBO measure of potential output	GMM
Christiano, Eihenbaum, and Evans (2005)	Not specified	system	VAR	GMM
Dib (2003)	Stationary AR	system	Linear trend	MLE
Smets and Wouters (2007)	Stationary AR	system	First difference	Bayesian
Smets and Wouters (2003)	Stationary AR	system	HP	Bayesian
Kim (2000)	Stationary AR	system	Linear trend	MLE
McGrattan, Rogerson, and Wright (1997)	Stationary AR	system	Linear trend and HP	MLE
Altug (1989)	Unit root	system	First differences	MLE in frequency domain
Fuhrer and Rudebusch (2004)	Not specified	equation	HP, one-sided BP, CBO, linear trend with breaks, quadratic deterministic trend	MLE, GMM
Fuhrer (1997)	Not specified	equation	HP, linear trend, quadratic trend	GMM
Kydland and Prescott (1982)	Permanent and transitory components	system	HP	calibration
Altig, Christiano, Eichenbaum, and Linde (2004)	Unit root with serial correlation in growth rates	system	First difference	GMM
Ireland (2004)	Unit root	system	Stationary variables and growth rates of nonstationary variables	MLE
Christiano and Eichenbaum (1992)	Unit root	system	HP	GMM
Burnside and Eichenbaum (1996)	Stationary AR	system	HP	GMM
Burnside, Eichenbaum and Rebelo (1993)	Stationary AR	system	HP	GMM

Table 2. Basic one-shock model.

DGP	ρ_z		(LT,$\hat{m}_t$) (1)	(HP,$\hat{m}_t$) (2)	(HP,$\tilde{m}_t$) (3)	(QD,$\Delta^{\rho}\hat{m}_t$) (4)	(FD,$\Delta^{1}\hat{m}_t$) (5)	(FD,$\Delta\hat{m}_t$) (6)	(HP,$\breve{m}_t$) (7)
			$\hat{\alpha}$: capital intensity, $\alpha = 0.33$						
DT	0.50	mean	0.3467	0.4201	0.9900	0.3435	0.3880	0.3330	0.3318
		sd	0.0264	0.0066	0.0000	0.0265	0.0162	0.0171	0.0203
		median	0.3462	0.4207	0.9900	0.3438	0.3866	0.3293	0.3312
DT	0.95	mean	0.4554	0.6245	0.9900	0.3260	0.4227	0.3552	0.3401
		sd	0.0996	0.0090	0.0000	0.0414	0.0596	0.0819	0.0446
		median	0.4500	0.6250	0.9900	0.3246	0.4136	0.3309	0.3372
DT	0.99	mean	0.7998	0.7430	0.9900	0.3241	0.3829	0.3607	0.3418
		sd	0.1931	0.0133	0.0000	0.0407	0.0850	0.0935	0.0508
		median	0.8835	0.7434	0.9900	0.3228	0.3591	0.3307	0.3373
ST	1.00	mean	0.9525	0.7804	0.9900	0.3178	0.3545	0.3499	0.3416
		sd	0.1370	0.0135	0.0000	0.0362	0.0886	0.0883	0.0515
		median	0.9747	0.7804	0.9900	0.3210	0.3284	0.3285	0.3375
			$\hat{\sigma}$: st.dev. of shocks to technology, $\sigma = 0.1$						
DT	0.50	mean	0.1048	0.1237	0.0037	0.1033	0.1840	0.1009	0.1005
		sd	0.0091	0.0064	0.0003	0.0086	0.0087	0.0101	0.0084
		median	0.1039	0.1226	0.0037	0.1024	0.1836	0.0990	0.1002
DT	0.95	mean	0.1076	0.1757	0.0047	0.0985	0.1380	0.1095	0.1002
		sd	0.0178	0.0121	0.0005	0.0124	0.0209	0.0309	0.0087
		median	0.1039	0.1753	0.0047	0.0977	0.1338	0.1004	0.0999
DT	0.99	mean	0.3377	0.2413	0.0043	0.0990	0.1175	0.1101	0.1006
		sd	0.1660	0.0211	0.0005	0.0106	0.0259	0.0286	0.0092
		median	0.3429	0.2409	0.0043	0.0980	0.1094	0.1002	0.1002
ST	1.00	mean	3.3506	0.2749	0.0041	0.0976	0.1077	0.1062	0.1005
		sd	1.7781	0.0255	0.0004	0.0088	0.0239	0.0237	0.0091
		median	2.8516	0.2738	0.0041	0.0970	0.0991	0.0986	0.1000
			$\hat{\rho}_z$: persistence of shocks to technology						
DT	0.50	mean	0.4611	0.2441	1.0000	0.4809	1.0000	0.4942	0.4809
		sd	0.0651	0.0411		0.0549		0.0810	0.0546
		median	0.4650	0.2455		0.4860		0.4947	0.4838
DT	0.95	mean	0.9270	0.5319	1.0000	0.9449	1.0000	0.9508	0.9501
		sd	0.0266	0.0329		0.0140		0.0127	0.0012
		median	0.9350	0.5329		0.9464		0.9498	0.9501
DT	0.99	mean	0.9120	0.5049	1.0000	0.9899	1.0000	0.9897	0.9903
		sd	0.0621	0.0281		0.0045		0.0055	0.0009
		median	0.9218	0.5070		0.9906		0.9896	0.9901
ST	1	mean	0.6932	0.4871	1.0000	0.9993	1.0000	0.9981	0.9999
		sd	0.0910	0.0257		0.0020		0.0026	0.0002
		median	0.6724	0.4890		1.0000		0.9991	1.0000

Note: This table presents summary statistics for estimates of $\alpha = 0.33$, $\sigma = 0.1$, and $\rho_z = (0.5, 0.95, 0.99, 1)$. The number of simulations is 1000. Sample size is T=300. In the top-row label (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables. LT is linear detrending, HP is Hodrick-Prescott filter, FD is first differencing, QD is quasi-differencing. Δ^1 denotes the restriction $\rho_z = 1$ when the model is solved in first differences. $\Delta^{\rho} = 1 - \rho_z L$ denotes quasi-differencing.

Table 3. Basic one-shock model with habit formation in consumption.

DGP	ρ_z		$(\text{LT},\hat{m}_t)$ (1)	$(\text{HP},\hat{m}_t)$ (2)	$(\text{HP},\tilde{m}_t)$ (3)	$(\text{QD},\Delta^{\rho}\hat{m}_t)$ (4)	$(\text{FD},\Delta^{1}\hat{m}_t)$ (5)	$(\text{FD},\Delta\hat{m}_t)$ (6)	$(\text{HP},\breve{m}_t)$ (7)
			$\hat{\alpha}$: capital intensity, $\alpha = 0.33$						
DT	0.95	mean	0.3991	0.7916	0.9647	0.3280	0.4399	0.3336	0.3389
		sd	0.0599	0.0441	0.0202	0.0248	0.0196	0.0260	0.0355
		median	0.3923	0.7980	0.9687	0.3262	0.4380	0.3336	0.3370
DT	0.99	mean	0.6067	0.8224	0.9836	0.3356	0.3582	0.3394	0.3389
		sd	0.1988	0.1216	0.0090	0.0252	0.0291	0.0295	0.0355
		median	0.6159	0.8630	0.9879	0.3366	0.3633	0.3386	0.3370
ST	1.00	mean	0.8061	0.8357	0.9864	0.3387	0.3386	0.3406	0.3399
		sd	0.1831	0.1705	0.0079	0.0282	0.0362	0.0288	0.0410
		median	0.8876	0.9008	0.9896	0.3365	0.3422	0.3358	0.3373
			$\hat{\sigma}$: st.dev. of shocks to technology, $\sigma = 0.1$						
DT	0.95	mean	0.0881	0.1758	0.0066	0.0972	0.1171	0.1009	0.9499
		sd	0.0132	0.0268	0.0023	0.0083	0.0095	0.0051	0.0009
		median	0.0868	0.1808	0.0062	0.0967	0.1164	0.1008	0.9498
DT	0.99	mean	0.0630	0.1271	0.0042	0.0984	0.1028	0.1011	0.9499
		sd	0.0179	0.1266	0.0011	0.0055	0.0052	0.0048	0.0009
		median	0.0608	0.0852	0.0038	0.0986	0.1024	0.1009	0.9498
ST	1.00	mean	0.0640	0.0452	0.0038	0.0997	0.1003	0.1013	0.9999
		sd	0.0330	0.0617	0.0010	0.0048	0.0046	0.0045	0.0001
		median	0.0622	0.0368	0.0035	0.0996	0.1004	0.1010	1.0000
			$\hat{\rho}_z$: persistence of shocks to technology						
DT	0.95	mean	0.9434	0.6450	1.0000	0.9453	1.0000	0.9485	0.0993
		sd	0.0070	0.0670		0.0103		0.0088	0.0079
		median	0.9449	0.6431		0.9460		0.9496	0.0992
DT	0.99	mean	0.9870	0.8058	1.0000	0.9875	1.0000	0.9879	0.0993
		sd	0.0040	0.1809		0.0055		0.0071	0.0079
		median	0.9881	0.8565		0.9886		0.9891	0.0992
ST	1	mean	0.9908	0.9439	1.0000	0.9980	1.0000	0.9962	0.0995
		sd	0.0077	0.0705		0.0033		0.0052	0.0079
		median	0.9926	0.9601		0.9996		0.9983	0.0994
			$\hat{\phi}$: habit formation in consumption, $\phi = 0.8$						
DT	0.95	mean	0.8280	0.9262	0.8255	0.7981	0.7536	0.8032	0.8033
		sd	0.0218	0.0757	0.2003	0.0091	0.0219	0.0216	0.0181
		median	0.8275	0.9336	0.8952	0.7979	0.7545	0.8033	0.8036
DT	0.99	mean	0.8386	0.7749	0.8518	0.8025	0.7960	0.8074	0.8033
		sd	0.1763	0.4104	0.1799	0.0150	0.0148	0.0241	0.0181
		median	0.8906	0.9362	0.9082	0.8047	0.7986	0.8059	0.8036
ST	1	mean	0.2864	0.7555	0.8221	0.8042	0.8033	0.8115	0.8037
		sd	0.3918	0.3366	0.2081	0.0140	0.0205	0.0219	0.0232
		median	0.2892	0.9515	0.8899	0.8032	0.8070	0.8060	0.8041

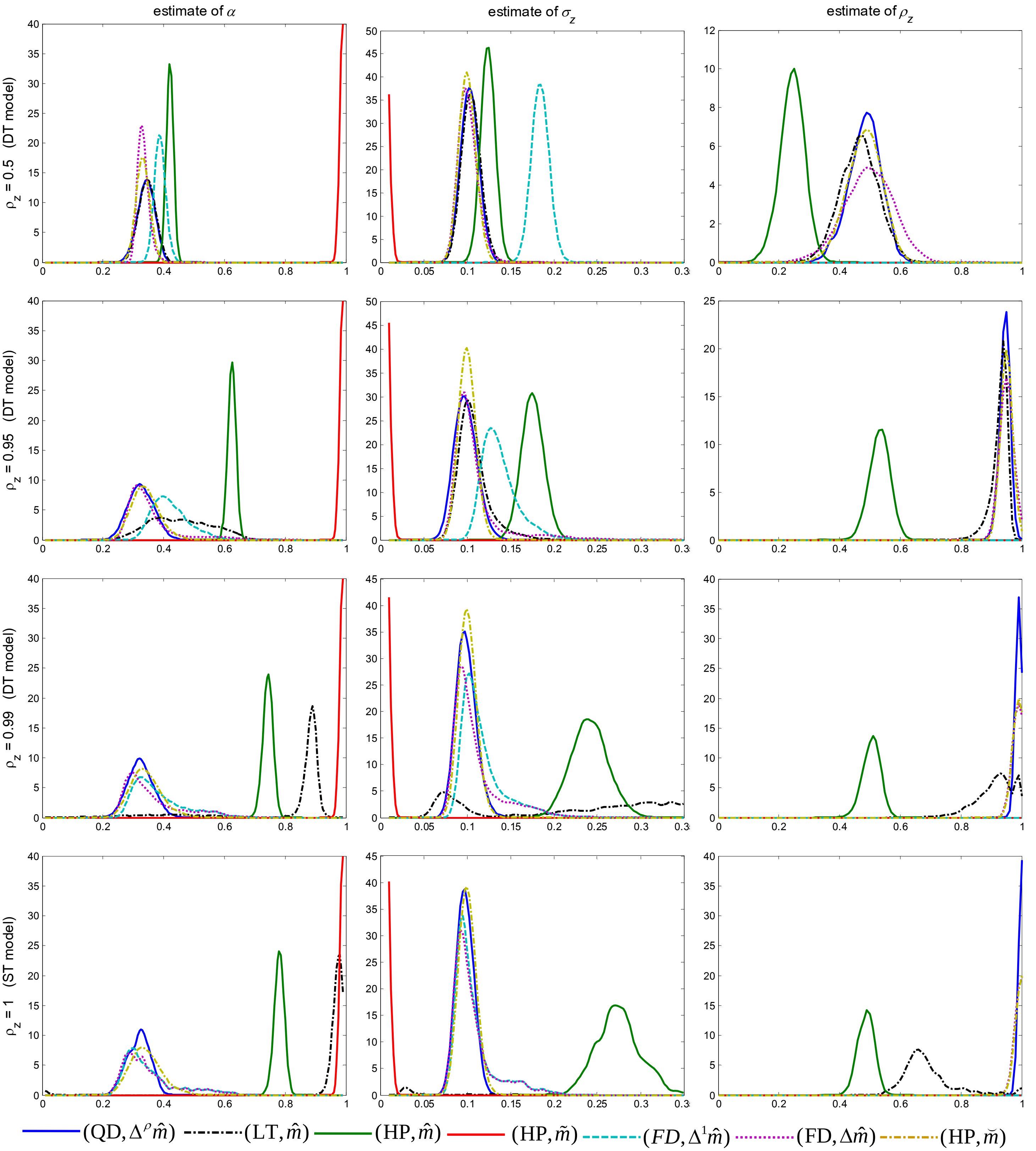
Note: This table presents summary statistics for estimates of $\alpha = 0.33$, $\sigma = 0.1$, and $\rho_z = (0.95, 0.99, 1)$. The number of simulations is 1000. Sample size is T=300. In the top-row label (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables. LT is linear detrending, HP is Hodrick-Prescott filter, FD is first differencing, QD is quasi-differencing. Δ^1 denotes the restriction $\rho_z = 1$ when the model is solved in first differences. $\Delta^{\rho} = 1 - \rho_z L$ denotes quasi-differencing.

Table 4. Two-shock model, estimate of α .

DGP	ρ_z		$(LT, \hat{m}_t)$ (1)	$(HP, \hat{m}_t)$ (2)	$(HP, \tilde{m}_t)$ (3)	$(QD, \Delta^{\rho} \hat{m}_t)$ (4)	$(FD, \Delta^1 \hat{m}_t)$ (5)	$(FD, \Delta \hat{m}_t)$ (6)	$(HP, \breve{m}_t)$ (7)
					σ_q = 0.025				
DT	0.95	mean	0.4520	0.6516	0.4487	0.3385	0.2911	0.3573	0.3328
		sd	0.1054	0.0278	0.1392	0.0285	0.0453	0.0766	0.0398
		median	0.4394	0.6510	0.4043	0.3310	0.2789	0.3326	0.3301
DT	0.99	mean	0.7798	0.7384	0.8887	0.3444	0.3499	0.3667	0.3320
		sd	0.1964	0.0323	0.1863	0.0356	0.0816	0.0891	0.0433
		median	0.8552	0.7394	0.9840	0.3326	0.3217	0.3360	0.3291
ST	1.00	mean	0.9197	0.7619	0.9497	0.3448	0.3622	0.3644	0.3350
		sd	0.1207	0.0326	0.1208	0.0359	0.0863	0.0865	0.0436
		median	0.9473	0.7626	0.9859	0.3330	0.3292	0.3312	0.3322
					σ_q = 0.05				
DT	0.95	mean	0.4442	0.6323	0.4031	0.3354	0.2900	0.3491	0.3244
		sd	0.1018	0.0393	0.0361	0.0194	0.0225	0.0556	0.0290
		median	0.4315	0.6279	0.3974	0.3316	0.2856	0.3334	0.3234
DT	0.99	mean	0.7489	0.6914	0.4546	0.3376	0.3269	0.3507	0.3231
		sd	0.2020	0.0425	0.0512	0.0207	0.0356	0.0594	0.0258
		median	0.8362	0.6896	0.4450	0.3332	0.3178	0.3322	0.3232
ST	1.00	mean	0.8794	0.7065	0.4725	0.3422	0.3432	0.3561	0.3265
		sd	0.1463	0.0436	0.0606	0.0246	0.0407	0.0578	0.0250
		median	0.9292	0.7047	0.4602	0.3374	0.3340	0.3377	0.3263
					σ_q = 0.15				
DT	0.95	mean	0.3813	0.5722	0.5378	0.3354	0.3225	0.3417	0.3232
		sd	0.0581	0.0471	0.0434	0.0123	0.0115	0.0310	0.0201
		median	0.3735	0.5679	0.5362	0.3331	0.3208	0.3311	0.3228
DT	0.99	mean	0.4182	0.5878	0.5481	0.3383	0.3316	0.3455	0.3263
		sd	0.1270	0.0446	0.0435	0.0125	0.0120	0.0298	0.0182
		median	0.3761	0.5846	0.5453	0.3363	0.3302	0.3351	0.3268
ST	1.00	mean	0.4433	0.5900	0.5512	0.3399	0.3344	0.3469	0.3276
		sd	0.1640	0.0445	0.0431	0.0127	0.0120	0.0291	0.0180
		median	0.3759	0.5866	0.5483	0.3380	0.3329	0.3371	0.3281

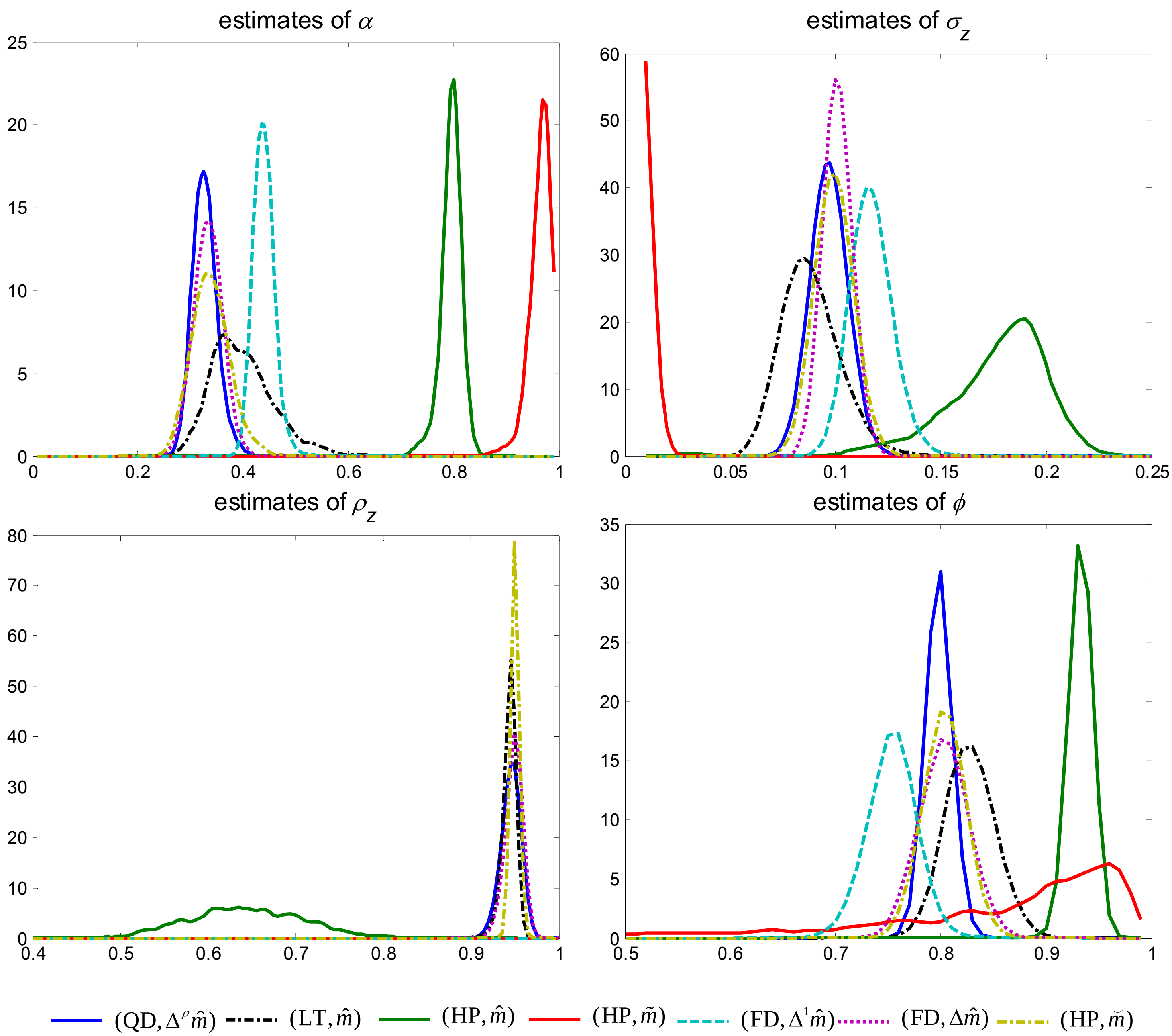
Note: Other parameters are fixed at $\alpha = 0.33, \rho_q = 0.8, \sigma_z = 0.1$. The number of simulations is 1000. Sample size is T=300. In the top-row label (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables. LT is linear detrending, HP is Hodrick-Prescott filter, FD is first differencing, QD is quasi differencing. Δ^1 denotes the restriction $\rho_z = 1$ when the model is solved in first differences. $\Delta^{\rho} = 1 - \rho_z L$ denotes quasi-differencing.

Figure 1. Kernel density of estimates, baseline model.



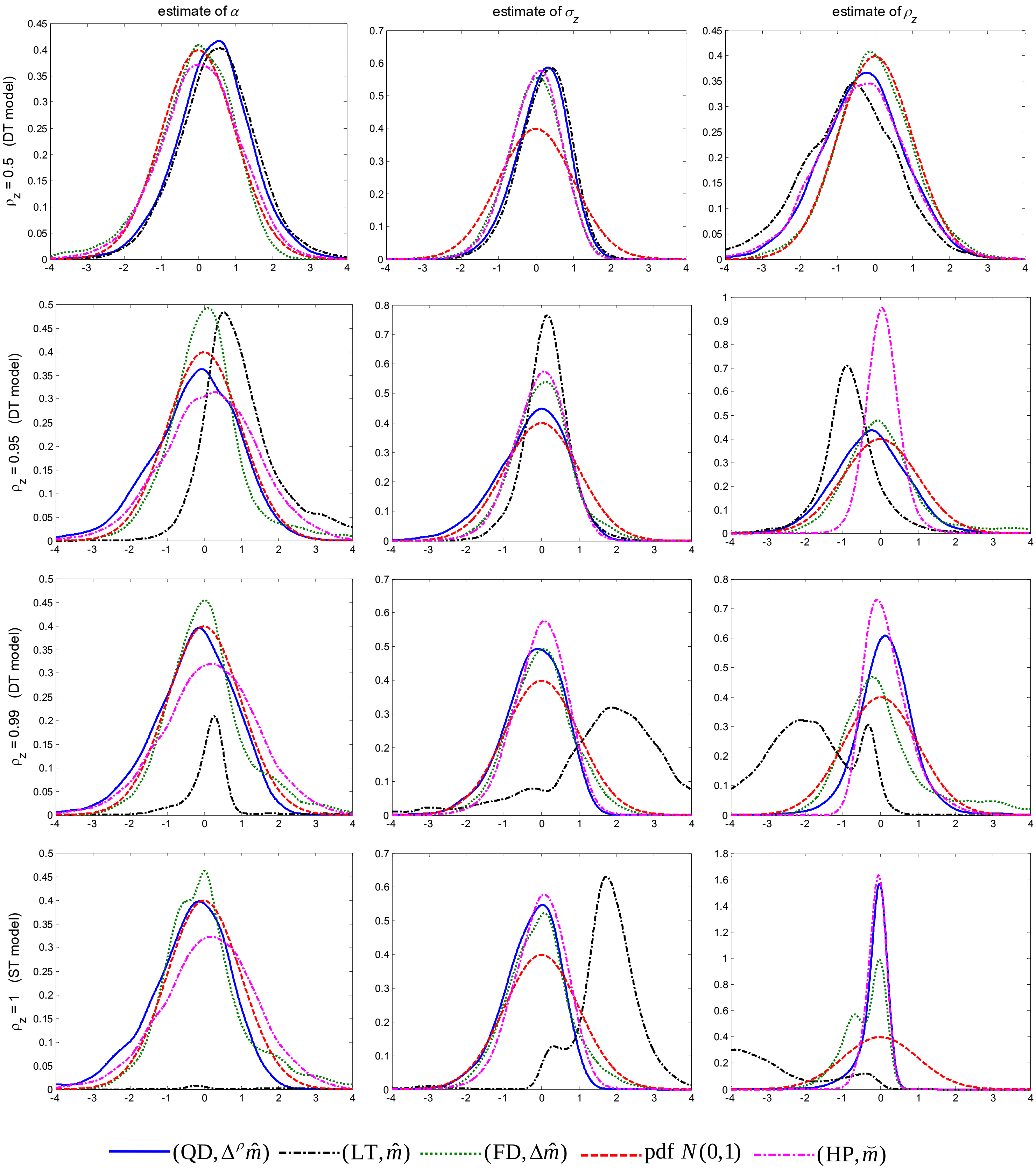
Note: This figure plots kernel density of $(\hat{\alpha}, \hat{\rho}, \hat{\sigma})$ generated in 1000 simulations. Bandwidth is 0.01. For the case presented in this figure the true values are $\alpha = 0.33, \sigma = 0.1$. True value of ρ_z is indicated on the left of the figure. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.

Figure 2. Kernel density of estimates for the model with habit formation, $\rho_z = 0.95$.



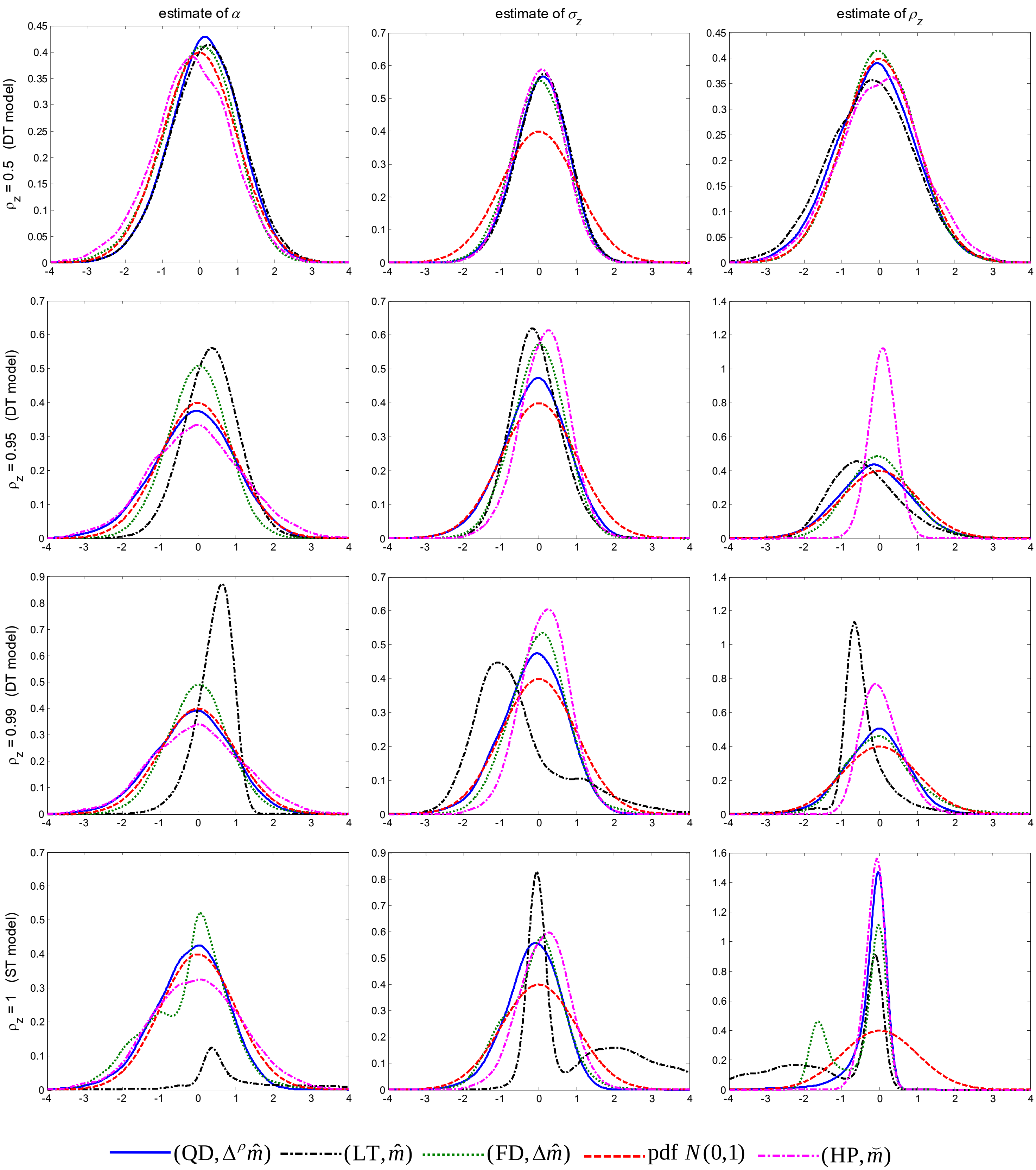
Note: This figure plots kernel density of $(\hat{\alpha}, \hat{\rho}, \hat{\sigma}, \hat{\phi})$ generated in 1000 simulations. Bandwidth is 0.01. For the case presented in this figure the true values are $\alpha = 0.33, \rho_z = 0.95, \phi = 0.8, \sigma = 0.1$. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.

Figure 3. Kernel density for t-statistic, baseline model, T=300.



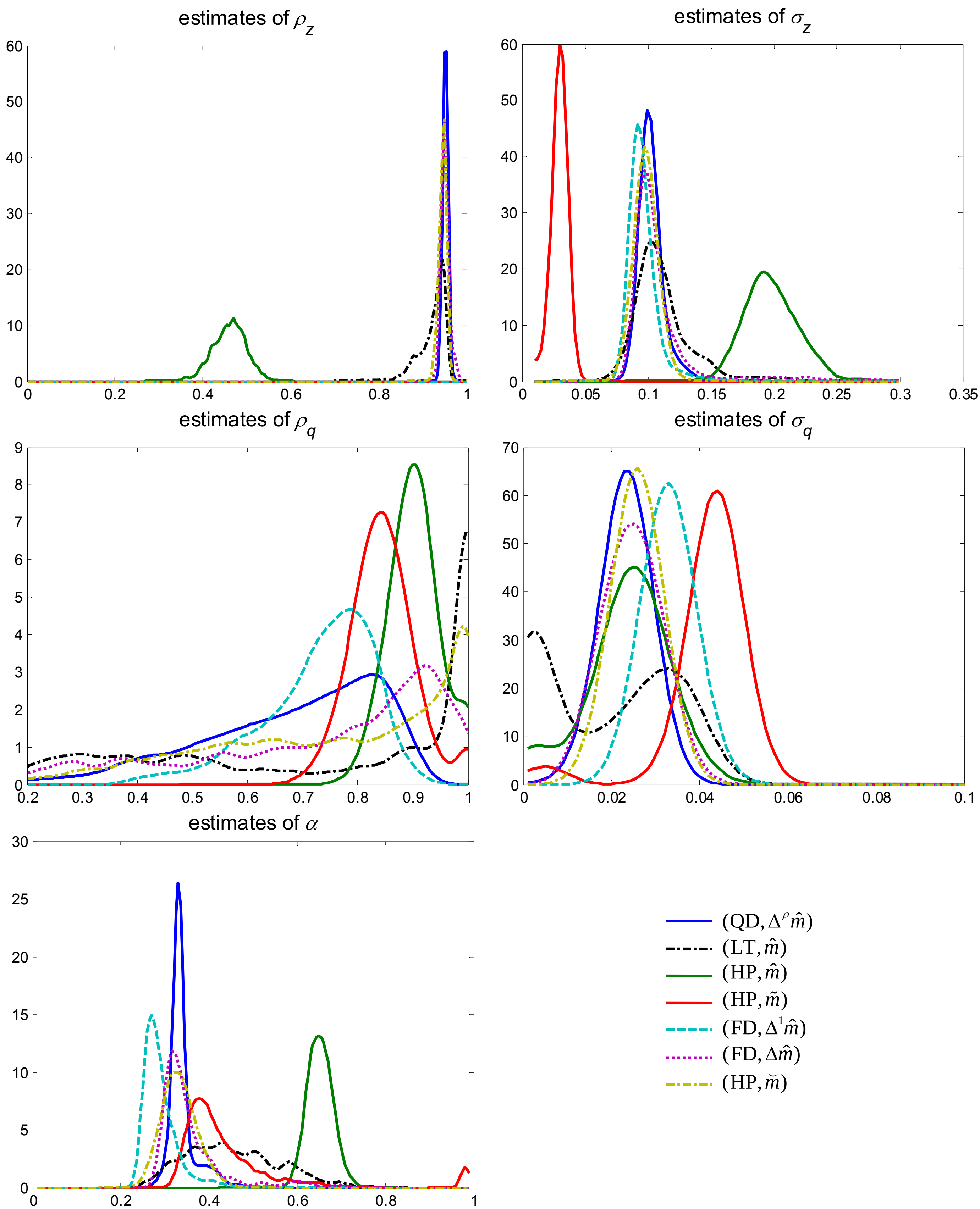
Note: This figure plots kernel density of t-statistic for $(\hat{\alpha}, \hat{\rho}, \hat{\sigma})$ generated in 1000 simulations. Bandwidth is 0.01. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.

Figure 4. Kernel density for t-statistic, baseline model, T=2000.



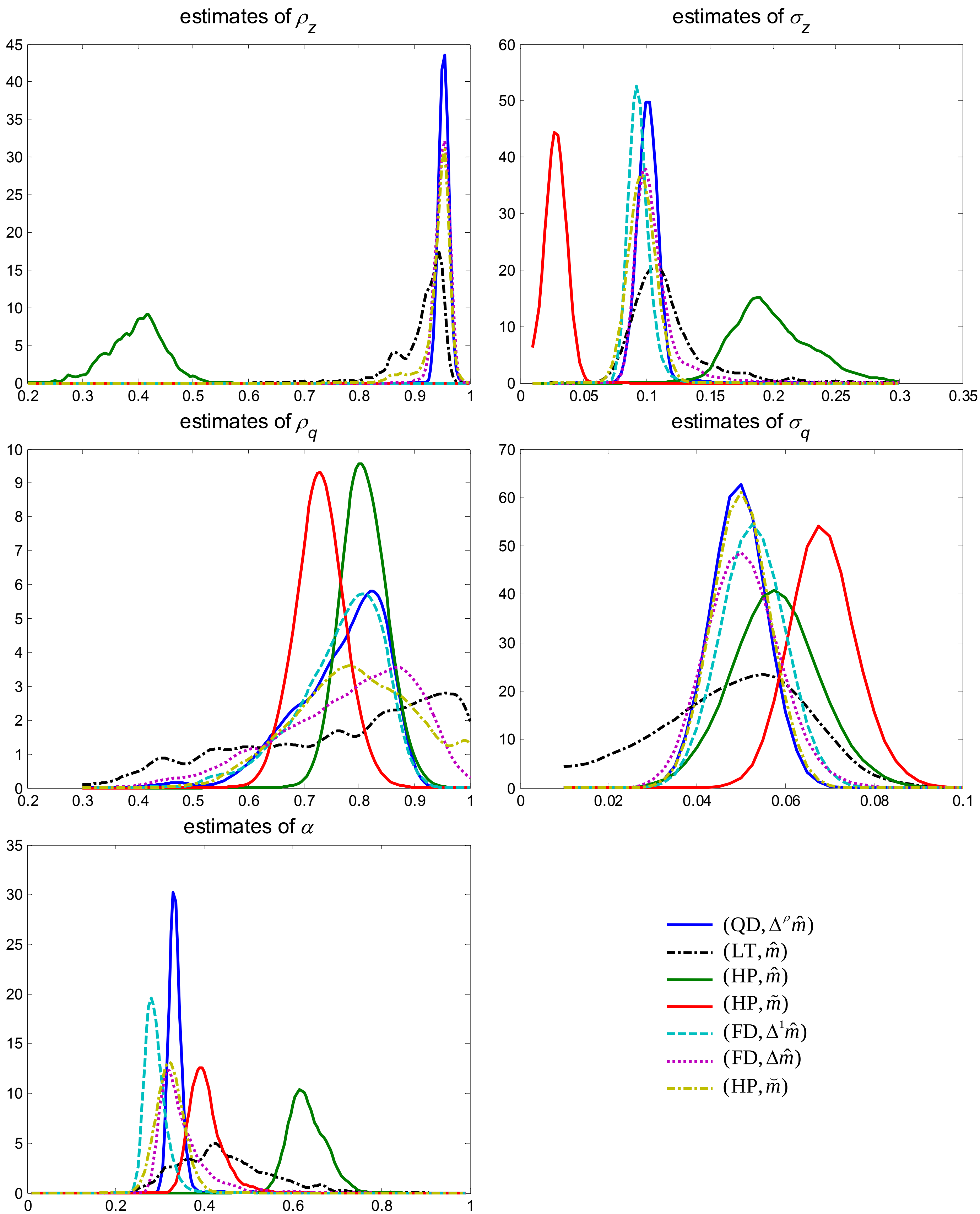
Note: This figure plots kernel density of t-statistic for $(\hat{\alpha}, \hat{\rho}, \hat{\sigma})$ generated in 1000 simulations. Bandwidth is 0.01. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.

Figure 5. Kernel density of estimates for the model with shocks to hours and technology, $\rho_z = 0.95$, $\sigma_q = 0.025$.



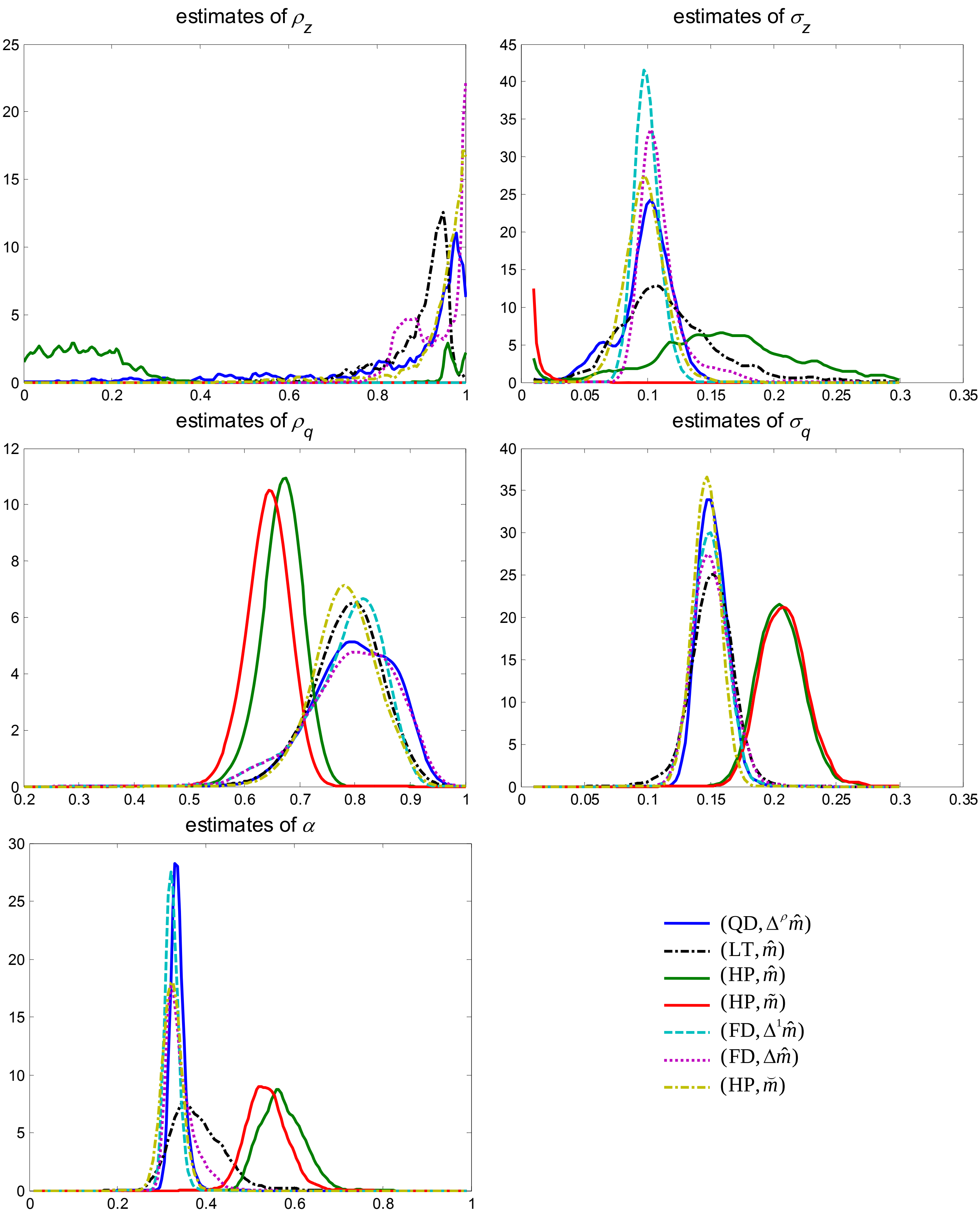
Note: This figure plots kernel density of $(\hat{\alpha}, \hat{\rho}_z, \hat{\sigma}_z, \hat{\rho}_q, \hat{\sigma}_q)$ generated in 1000 simulations. Bandwidth is 0.01. For the case presented in this figure the true values are $\alpha = 0.33, \rho_z = 0.95, \sigma_z = 0.1, \rho_q = 0.8, \sigma_z = 0.025$. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.

Figure 6. Kernel density of estimates for the model with shocks to hours and technology, $\rho_z = 0.95$, $\sigma_q = 0.05$.



Note: This figure plots kernel density of $(\hat{\alpha}, \hat{\rho}_z, \hat{\sigma}_z, \hat{\rho}_q, \hat{\sigma}_q)$ generated in 1000 simulations. Bandwidth is 0.01. For the case presented in this figure the true values are $\alpha = 0.33, \rho_z = 0.95, \sigma_z = 0.1, \rho_q = 0.8, \sigma_z = 0.05$. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.

Figure 7. Kernel density of estimates for the model with shocks to hours and technology, $\rho_z = 0.95$, $\sigma_q = 0.15$.



Note: This figure plots kernel density of $(\hat{\alpha}, \hat{\rho}_z, \hat{\sigma}_z, \hat{\rho}_q, \hat{\sigma}_q)$ generated in 1000 simulations. Bandwidth is 0.01. For the case presented in this figure the true values are $\alpha = 0.33, \rho_z = 0.95, \sigma_z = 0.1, \rho_q = 0.8, \sigma_z = 0.15$. In the legend (XX,YY), XX denotes the method of detrending and YY indicates the model concept of the observed variables.