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Abstract

Common affine term structure models (ATSMs) suggest that bond yields include both expected short rates and term premiums, in violation of the strictest forms of the expectations hypothesis (EH). Similarly, forward foreign exchange contracts likely include not only expected depreciation but also a sizeable premium, which similarly contradicts pure interest rate parity (IRP) and complicates inferences about anticipated returns on foreign currency exposure. Closely following the underlying logic of ubiquitous term structure models in parallel, and rather than the usual econometric approach, this study derives arbitrage-free affine forward currency models (AFCMs) with closed-form expressions for both unobservable variables. Model calibration to eleven forward U.S. dollar currency pair term structures, and notably without any information from corresponding term structures, from the mid-to-late 1990s through early 2015 fits the data closely and suggests that the premium is indeed nonzero and variable, but not to the degree implied by previous econometric studies.

Key words: arbitrage-free model, foreign exchange

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1. Introduction

In addition to projected variance and covariance among exchange rates, information about expected returns on currency positions is essential for global financial asset allocation. Forward contracts embed investors' expectations for depreciation and therefore anticipated returns on currency positions. But the problem is that, as a gargantuan literature documents (e.g., Froot and Thaler, 1990; Froot, 1990; Engel, 1996), forward quotes also include a sizeable and perhaps anomalous premium. Using observed quotes and subsequent interest rates as well as forward exchange rates, several studies find that in violation of uncovered and sometimes covered interest rate parity (UIRP and CIRP), currencies with higher interest rates tend to appreciate rather than to depreciate, as carry trades are persistently profitable on average. The standard approach is to consider regressions that broadly resemble

$$\ln\left(\frac{E_t}{E_0}\right)=\alpha+\beta\left(y_{0,\mathrm{t}}^{\$}-y_{0,\mathrm{t}}^{€}\right)+\eta \tag{1}$$

where E is the local price of foreign exchange, $y_{0,t}^{\$}-y_{0,t}^{€}$ represents the relevant yield differential over horizon 0 through t, and $\beta=1$ as well as $\alpha=0$ are the null hypotheses that the literature resoundingly rejects—β is nearly always less than one and is frequently negative. This so-called "forward discount anomaly" might reflect a fair ex ante premium for such trades or chronically incorrect market forecasts. Either way, any attempt to read expectations from forwards should make some allowances for possible premiums.

Rather than start from this ubiquitous econometric approach, what follows takes a different tact, draws from continuous time-finance in general as well as interest rate models in particular, and outlines arbitrage-free affine forward currency models (AFCMs) to estimate both unobservable quantities, notably ex ante. As discussed below, despite the assumption that IRP holds in the short run, the models produce an explicit expression for the (risk-neutral) forward discount premium, defined as the difference between the model-implied forward rate and the expected depreciation rate. A key analogy with affine term structure models (ATSMs) is that model-based yields comprise expected short rates and (risk-neutral) term premiums, even in the absence of instantaneous arbitrage opportunities along the yield curve (e.g., Vasicek, 1977; Langetieg, 1980).

Few if any existing studies exploit the parallels. For example, Backus et al. (2001) examine whether ATSMs produce forward rates that are consistent with the two key findings in Fama (1984) that expected depreciation and the premium are negatively correlated and that the variance of the premium is greater than the variance of expected depreciation. They find that ATSMs are inconsistent with the forward premium anomaly unless either nominal interest rates are negative or the underlying state variables have asymmetric effects on state prices. In contrast, this study uses no information from term structures and models forward exchange rate quotes directly to extract the premium from expected depreciation, with no restrictions on the parameters to conform to Fama (1984).

Besides isolation of the discount premium in closed-form solutions, calibration to forward term structures of 11 $U.S. currency pairs from the mid- to late-1990s through early 2015 suggests that the premium is indeed not only non-zero but also both spatially and temporally variable. However, the degree of variance over time differs across pairs and in general is lower than some econometric studies suggest (e.g., Fama, 1984; Hodrick and Srivastava, 1986). In short, these results in no way reconcile the forward premium anomaly per se, but AFCM-implied depreciation rates might include useful information about expected returns to foreign currency positions.

The next section outlines some rudimentary concepts. Then, to isolate the discount premium, the following section derives a single-factor model with perhaps the narrowest possible assumptions regarding the stochastic dynamics of instantaneous depreciation. Next, the discussion describes an extension for calibration and application to latent factors, time-varying premiums, and an underlying multi-factor Gaussian random process. The remaining sections describe the parameter estimation of the Gaussian model and the general empirical results.

2. Some Rudiments on Interest Rate Parity and Parallels with Term Structure Models

Define Q as the depreciation (or appreciation) over a non-instantaneous period from time 0 to t for the domestic price of a foreign currency, E, as in

$$E_t = E_0 \exp\left[Qt\right] \tag{2}$$

For reference, the corresponding expression for the price of a zero-coupon bond is

$$P_t = P_0 \exp\left[yt\right] \tag{3}$$

where y is the continuously-compounded yield on the bond P that matures at t, and the left-hand-side is equal to par at expiry. Returning to foreign exchange, broadly analogous to the pure expectations hypothesis (EH) of interest rates, which suggests that longer-dated yields represent the average of expected short rates, r, consider Q as the mean of instantaneous depreciation rates, q, over the initial time 0 to t, as in

$$y \approx \frac{1}{t}\int_0^t \varepsilon\left\{r_s\right\}ds \Leftrightarrow Q \approx \frac{1}{t}\int_0^t \varepsilon\left\{q_s\right\}ds \tag{4}$$

where $\varepsilon\left\{\cdot\right\}$ is the expectations operator, and both expression ignore Jensen's inequality.

To review the motivation behind (1), interest rate parity (IRP) dictates that expected depreciation is equal to the interest rate differential between the two countries, which in turn can be decomposed under a relaxation of the EH into rate expectations and (time-varying) term premiums. There is substantial debate and a huge literature on whether UIRP or even CIRP holds, with very discouraging results from dozens of earlier studies (Froot and Thaler, 1990). However, some studies find that some data in the very short run are consistent with the theory (e.g., Chaboud and Wright, 2005), and other studies also report that UIRP holds to some degree in the long run (e.g., Chinn and Meredith, 2005). But regardless of whether ex post returns follow IRP, under the expectation of no arbitrage, investors cannot take a short position in a futures contract, borrow one unit of a foreign currency (say the dollar price of a euro, \$/€), at the domestic risk-free interest rate at the current rate of exchange, deposit that amount in a foreign-risk-free-rate-bearing account, and deliver the foreign-dominated funds at the forward exchange rate agreed upon at time 0 for expiry at *t*, $F_{0,t}$, for a profit. Rather, interest rate parity from borrowing a single unit of domestic currency implies

$$-\exp\left[y_{0,t}^{\$}t\right]+E_0^{-1}\exp\left[y_{0,t}^{€}t\right]F_{0,t}=0 \tag{5}$$

or

$$\ln\frac{F_{0,t}}{E_0}=\left(y_{0,t}^{\$}-y_{0,t}^{€}\right)t$$

Of course, substituting (2) for the right hand side suggests that IRP demands $y_{0,t}^{\$}-y_{0,t}^{€}=Q$, and the continuous-time equivalent along with (4) suggest that $r^{\$}-r^{€}=q$. Beyond the relation between the spot observed at 0 and the forward quote, consider the term structure of currency forwards priced at time 0 for the future period between $t-\Delta$ and t, say.[1] By extension, forwards must follow parity with respect to differentials in corresponding forward interest rates, f, as in

$$\ln\frac{F_{0,\mathrm{t}}}{F_{0,\mathrm{t}-\Delta}} = \left(f^{\$}_{0;\mathrm{t}-\Delta,\mathrm{t}} - f^{€}_{0;\mathrm{t}-\Delta t}\right)(t-\Delta) \tag{6}$$

Correspondingly, besides these static relations, the expected instantaneous change in a single forward currency contract with a fixed maturity t over time 0 to dt, scaled by its initial value under interest rate parity follows

$$\begin{aligned} \varepsilon\left\{d\ln\left(F_{\mathrm{dt,t}}\right)\right\} &= \left(f^{\$}_{\mathrm{dt,t}} - f^{€}_{\mathrm{dt},t}\right)dt \\ &\approx \varepsilon\left\{\frac{dF}{F}\right\} \\ &\approx \varepsilon\left\{\frac{F_{dt,t}-F_{0,t}}{F_{0,t}}\right\} \end{aligned} \tag{7}$$

One might argue that these arbitrage conditions comprise risk neutral relations that preclude any premia, given that by definition arbitrage is a riskless trade. However, within the IRP framework, the right hand side of, say, (6) can also be decomposed to relax the pure EH of interest rates. Accordingly, the log difference in forward exchange rates between $t-\Delta$ and t is the sum of the difference in (instantaneous) anticipated short rates and the difference in the term premiums, ρ, investors require in the two countries during that interval, following

$$\ln\frac{F_{0,\mathrm{t}}}{F_{0,\mathrm{t}-\Delta}} = \left[\left(\varepsilon\left\{r^{\$}_{0;t-\Delta,t}\right\} - \varepsilon\left\{r^{€}_{0;t-\Delta,t}\right\}\right) + \left(\rho^{\$}_{0;t-\Delta,t} - \rho^{€}_{0;t-\Delta,t}\right)\right](t-\Delta) \tag{8}$$

An ATSM-based decomposition of forward rates across the two markets, which rules out arbitrage along the two curves, implies anticipated depreciation and a currency premium between the pair (equivalent to the spread in term premiums), with the further assumption of IRP.

[1] Even if one argues (under the assumption of constant longer-run expectations) that the forward for time t contains no information about the change in the spot from 0 to t, the same cannot be said along the term structure of forward quotes beyond 0, between $t-\Delta$ and t, say. The argument in the context of the following equation must imply that $F_{0,\mathrm{t}-\Delta}$ includes all relevant information, but indeed $F_{0,\mathrm{t}-\Delta}$ is of course a forward quote.

Is there is a way to glean these quantities exclusively from forward currency contracts, rather than estimate separate ATSMs across the term structure pairs? Such a method would isolate spreads in both expected short rates and term premiums from currency forwards, abstracting from the level of both quantities and using no information from underlying yield curves. The desired decomposition can be recast, in terms of currency forwards, as

$$\ln\frac{F_{0,\mathrm{t}}}{F_{0,\mathrm{t}-\Delta}} = \left(\varepsilon\left\{q_{0;t-\Delta,t}^{\$/€}\right\} + \rho_{0;t-\Delta,t}^{\$/€}\right)\left(t-\Delta\right) \tag{9}$$

Besides more direct estimation for foreign exchange, the motivation for using currency forwards instead of yield curves includes the fact that measurement of the latter is non-trivial. Government bond term structures, for example, might not be fully applicable to foreign exchange market participants. Currency unions such as EMU compound this issue because arguably no single sovereign issuer of risk-free securities is truly representative. Also, germane to the environment at the time of writing, estimation with currencies may circumvent problems associated with the lower nominal bound for interest rates, given the implied depreciation rates have no such limits.

To grasp some intuition behind $\rho_{0,t}^{\$/€}$, consider the expression for the ex post return on a €-funded carry trade from 0 to *t*, rx_t, as in

$$rx_t = y_{0,t}^{\$} - y_{0,t}^{€} - \ln\frac{E_t}{E_0} \tag{10}$$

Invoking UIRP with respect to the interest rate differential, taking expectations, and considering the decomposition in (9) follows

$$\varepsilon\left\{rx_t\right\} = \varepsilon\left\{\ln\frac{E_t}{E_0}\right\} + \rho_{0;t-\Delta,t}^{\$/€} - \varepsilon\left\{\ln\frac{E_t}{E_0}\right\}$$

which of course shows, given the cancelation of the first and third terms, that the premium is the expected excess return on the position. In addition, the expected carry trade return can be expressed under similar assumptions as the term premium spread,

$$\varepsilon\left\{rx_t\right\} = \varepsilon\left\{\frac{1}{t}\int_0^t r_s^{\$} ds\right\} - \varepsilon\left\{\frac{1}{t}\int_0^t r_s^{€} ds\right\} + \left(\rho_{0,t}^{\$} - \rho_{0,t}^{€}\right) - \varepsilon\left\{\ln\frac{S_t}{S_0}\right\}$$

where the first, second, and fifth terms sum to zero.

3. A Partial Differential Equation for Forward Currency Contracts

Now suppose that the instantaneous currency depreciation rate, q, is not deterministic as in the previous section but follows some stochastic process, say, for simplicity

$$dq = \mu dt + \sigma dW \tag{11}$$

where μ is a drift term, σ is the volatility parameter, and dW is a Brownian motion increment with $\varepsilon\left\{dW\right\} = 0$. By analogy, the risk-free interest rate process in the domestic and foreign currency would follow, respectively.

$$\begin{aligned} dr^{\$} &= \mu^{\$} dt + \sigma^{\$} dW^{\$} \\ dr^{€} &= \mu^{€} dt + \sigma^{€} dW^{€} \end{aligned}$$

The objective is to derive a formula for a forward contract, F, a financial claim on the future exchange rate. To be sure, multiple factors influence F, as indeed simple IRP suggests forward contracts are a function of domestic and foreign interest rates as well as the spot exchange rate, all of which are random variables. However, just to start, suppose that F is a function of time and the stochastic depreciation rate, $F\left(q,t\right)$. Again, by analogy, remedial single-factor short-rate models express the price of a bond as a function of time and the stochastic interest rate, $P^{\$}\left(r^{\$},t\right)$ (e.g., Brennan and Schwartz, 1977). Given Ito's lemma, (11), and some rearranging, the instantaneous "return," or more precisely the instantaneous change in the forward exchange value of the local currency scaled by its initial price, follows

$$\frac{dF}{F}=\frac{1}{F}\left(\frac{\partial F}{\partial t}+\mu\frac{\partial F}{\partial q}+\frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial q^2}\right)dt+\frac{\sigma}{F}\frac{\partial F}{\partial q}dW \tag{12}$$

Similar arguments produce the following expression for the instantaneous return on a risk-free bond, as in

$$\frac{dP^{\$}}{P^{\$}}=\frac{1}{P^{\$}}\left(\frac{\partial P^{\$}}{\partial t}+\mu^{\$}\frac{\partial P^{\$}}{\partial r^{\$}}+\frac{1}{2}\left(\sigma^{\$}\right)^2\frac{\partial^2 P^{\$}}{\partial\left(r^{\$}\right)^2}\right)dt+\frac{\sigma}{P^{\$}}\frac{\partial P^{\$}}{\partial r^{\$}}dW^{\$}$$

Now assume that the expected value of $\frac{dF}{F}$, which may conform to IRP, follows (7). That is, the drift term in (12) obeys the following partial differential equation (PDE), as in with simple rearranging

$$\begin{aligned}\varepsilon\left\{\frac{dF}{F}\right\}&=\frac{1}{F}\left(\frac{\partial F}{\partial t}+\mu\frac{\partial F}{\partial q}+\frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial q^2}\right)dt\\&=\left(r^{\$}-r^{€}\right)dt\\&=qdt\end{aligned} \tag{13}$$

Multiplying through by F and $1/dt$ and given that the interest rate parity assumption implies $r^{\$}-r^{€}=q$, the expression becomes

$$\frac{\partial F}{\partial t}+\mu\frac{\partial F}{\partial q}+\frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial q^2}=qF$$

To further specify the PDE, consider the market price per unit of risk with respect to instantaneous depreciation, λ, which follows

$$\frac{\varepsilon\left\{\frac{dF}{F}\right\}-\left(r^{\$}-r^{€}\right)}{\sqrt{VAR\left\{\frac{dF}{F}\right\}}}=\frac{\mu-q}{\sigma}=\lambda \tag{14}$$

Given (12), the risk-neutral PDE for the current forward the must follows

$$\frac{\partial F}{\partial t}+\left(\mu-\sigma\lambda\right)\frac{\partial F}{\partial q}+\frac{1}{2}\sigma^{2}\frac{\partial^{2}F}{\partial q^{2}}-qF=0 \tag{15}$$

There are further general similarities between the derivation of this PDE and the bond pricing equation that underpins ATSMs. Very briefly, the general argument in Vasicek (1977), Brennan and Schwartz (1977), and Langetieg (1980) is that a portfolio of bonds that has a deterministic return—given a unique hedge ratio—must earn the risk-free rate in the absence of arbitrage. For example, the expected return on a risk-free bond, $\varepsilon\left\{\frac{dP^{\$}}{P^{\$}}\right\}$, (analogous to the scaled change in the forward exchange rate) under the EH must follow

$$\varepsilon\left\{\frac{dP^{\$}}{P^{\$}}\right\}=\frac{1}{P^{\$}}\left(\frac{\partial P^{\$}}{\partial t}+\mu^{\$}\frac{\partial P^{\$}}{\partial r^{\$}}+\frac{1}{2}\left(\sigma^{\$}\right)^{2}\frac{\partial^{2}P^{\$}}{\partial\left(r^{\$}\right)^{2}}\right)dt=r^{\$}dt$$

And, the specification of the market price of risk, which maps risk-neutral pricing to the physical measure that in turn affords expressions for term premiums, produces the familiar equation (e.g., Vasicek, 1977)

$$\frac{\partial P^{\$}}{\partial t}+\left(\mu^{\$}-\sigma^{\$}\lambda^{\$}\right)\frac{\partial P^{\$}}{\partial r^{\$}}+\frac{1}{2}\left(\sigma^{\$}\right)^{2}\frac{\partial^{2}P^{\$}}{\partial\left(r^{\$}\right)^{2}}-r^{\$}P^{\$}=0$$

In this parallel application to foreign exchange, the underlying factor is not the short rate but the instantaneous depreciation rate.

4. A Solution for Forward Currency Contracts

Similar to single-factor, short-rate ATSMs, a proposed solution implies that the continuously-compounded return on the forward, Q, is a linear function of the instantaneous depreciation rate, q, as in

$$\begin{aligned} F_t &= E_0 \exp\left[Q(t,T)t\right] \\ &= E_0 \exp\left[A(t,T) + B(t,T)q\right] \end{aligned} \tag{16}$$

with the obvious affine analog for bond pricing, following

$$\begin{aligned} P_t^{\$} &= P_0^{\$} \exp\left[y^{\$}(t,T)t\right] \\ &= P_0^{\$} \exp\left[A^{\$}(t,T) + B^{\$}(t,T)r^{\$}\right] \end{aligned}$$

Instead of the terminal condition, say, for a bond that pays par at maturity (i.e., Vasicek, 1977, Brennan and Schwartz, 1977), the initial conditions stem from the simple requirement that the current instantaneous forward rate equals the current spot rate.[2] Furthermore, given that the equation must hold for all values of q, the conditions for the affine solution follow

$$\begin{aligned} F_0 = E_0 &= F_0 e\left[A(0,T) + B(0,T)q\right] \\ 0 &= A(0,T) = B(0,T) \end{aligned} \tag{17}$$

Now take the relevant partial derivatives from (16) with respect to the instantaneous depreciation rate as well as time and substitute the derivatives into the PDE, (15), as in

$$F\left(\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t}q\right) + (\mu - \sigma\lambda)FB + \frac{1}{2}\sigma^2 FB^2 - qF = 0 \tag{18}$$

With some further rearranging, the PDE reduces to a system of two tractable ordinary differential equations (ODEs), following

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[2] Note that the proposed affine solution conforms strictly to IRP under the conditions that $A = 0$, $B = t$, and $q = r^{\$} - r^{€}$. Similarly, the assumed solution for bonds that is consistent with the pure EH implies that $A^{\$} = 0$ and $B^{\$} = t$. However, the fact that common calibrations of the former from ATSMs do not produce these conditions (but instead produce non-zero term premiums in the data) does not violate the no arbitrage condition underlying the model. Instead, those assumptions lead to the derivation of the partial differential equations.

$$-\frac{\partial B}{\partial t}+1=0 \tag{19}$$

and

$$\frac{\partial A}{\partial t}+(\mu-\sigma\lambda)B+\frac{1}{2}\sigma^2B^2=0 \tag{20}$$

The solution for (19) given the initial condition defined in (17) follows

$$B(t,T)=t \tag{21}$$

And, given substitution of (21) into (20), simple integration with respect to *t*, and again the relevant initial condition in (17), the solution to the second ODE is

$$A(t,T)=-\frac{1}{2}(\mu-\sigma\lambda)t^2-\frac{1}{6}\sigma^2t^3 \tag{22}$$

With (21) and (22), the solution to the PDE for the forward contract is

$$F_t=F_0\exp\left[-\frac{1}{2}(\mu-\sigma\lambda)t^2-\frac{1}{6}\sigma^2t^3+qt\right] \tag{23}$$

And, following (16), the corresponding expression for the depreciation rate over *t* follows

$$Q_t=q-\frac{1}{2}(\mu-\sigma\lambda)t-\frac{1}{6}\sigma^2t^2 \tag{24}$$

which traces out a term structure of depreciation rates, with an initial value of *q*.

5. The Arbitrage-free-implied Currency Premium

To extract any implied premium from the model, ρ_{FX}, and therefore expected returns, the relevant question is whether the instantaneous forward depreciation rate from the solution, $q_{f,t}$, at some future date *t* is an unbiased predictor of the expected future instantaneous depreciation rate, $E\{q_t\}$. To start, the premium defined formally is

$$\rho_{FX} = q_{f,t} - E\{q_t\} \tag{25}$$

and forward depreciation rate for the discrete interval from t to $t + \varepsilon$ follows

$$q(t, t+\Delta) = \frac{1}{\Delta} \ln \frac{F_{t+\Delta}}{F_t} \tag{26}$$

The instantaneous forward depreciation rate at t then follows

$$q_f = \lim_{\Delta \to 0} \frac{\ln F_{t+\Delta} - \ln F_t}{\Delta} = \frac{\partial \ln F}{\partial t} \tag{27}$$

Therefore, using the solution of the model, (23),

$$q_{f,t} = q - (\mu - \sigma\lambda)t - \frac{1}{2}\sigma^2 t^2 \tag{28}$$

Now, to determine $E\{q_t\}$, given the stochastic process of the depreciation rate from (11), the expected value of the forward instantaneous depreciation follows

$$E\left\{\int_0^t dq\right\} = E\left\{\mu\int_0^t d\tau + \sigma\int_0^t dW_\tau\right\} \Leftrightarrow E\{q_t\} = q + \mu t \tag{29}$$

Given that the premium is the difference between the model-implied forward depreciation, (28), and the expected value, (29), (25) becomes

$$\rho_{FX} = -2\mu t - \sigma\lambda t - \frac{1}{2}\sigma^2 t^2 \tag{30}$$

which is indeed non-zero unless the following condition holds

$$\mu = -\frac{1}{2}\sigma\lambda - \frac{1}{4}\sigma^2 t \tag{31}$$

In addition, again the model assumes that the instantaneous deterministic drift must follow IRP, but unless (31) holds, then the model also is flexible enough to allow for non-zero forward premium. Therefore, consistent with the preponderance of econometric evidence (Engle, 1996), the broad implication is that forward quotes are not unbiased expectations of depreciation, perhaps just as forward interest rates are not unbiased expectations of short rates.

6. A Multi-Factor Gaussian AFCM: Derivation and Estimation

Indeed, calibration should disentangle the premium from market expectations for returns, but however illustrative, the model outlined previously is too restrictive for application. Consider instead a latent factor approach with time-varying forward premiums as well as an mean-reverting alternative to the stochastic process in (11), following

$$\underset{n\times 1}{dX_t} = \underset{n\times n}{\kappa}\left(\theta - \underset{n\times 1}{X_t}\right)dt + \underset{n\times n}{\Sigma}\,\underset{n\times 1}{dW_t} \tag{32}$$

where X_t is an $n\times 1$ vector of underlying factors, θ is $n\times 1$, κ is an $n\times n$ lower-triangular matrix, Σ is a diagonal $n\times n$ matrix,[3] and dW is an $n\times 1$ vector of Gaussian disturbances. The instantaneous depreciate rate, q, is a linear function of the factors, as in

$$q_t = \delta_0 + \delta_1^T X_t \tag{33}$$

where δ_0 is a scalar, and δ_1 is $n\times 1$. Also, the vector of market prices of risk is a linear function of the factors, following

$$\lambda_t = \lambda_0 + \lambda_1^T X_t \tag{34}$$

where λ_o is $n\times 1$, and λ_1 is $n\times n$. Similar to the single-factor case, arbitrage implies a matrix PDE for the futures contract that follows

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}^T\left[\kappa\left(\theta - X_t\right) - \Sigma\left(\lambda_0 + \lambda_1^T X_t\right)\right] + \frac{1}{2}\left(\Sigma\frac{\partial F}{\partial X}\right)^T - \left(\delta_0 + \delta_1^T X_t\right)F = 0 \tag{35}$$

The affine form of the proposed solution to the PDE follows

[3] Different normalizations are of course possible, such as a diagonal κ and a lower-triangular Σ. See Dai and Singleton (2000). Not also that in the estimation the n elements of λ_1 (θ) are equal to 1 (0).

$$F(X,t) = E_0 \exp\left[Q(t,T)t\right] = E_0 \exp\left[A(t,T) + B(t,T)^T X_t\right] \tag{36}$$

And, following the steps and notation outlined in the Appendix A, the relevant solutions follow

$$B(t,T) = \left(\kappa^{*T}\right)^{-1}\left(I - \exp\left[-\kappa^{*T}t\right]\right)\delta_1 \tag{37}$$

and

$$A(t,T) = \delta_0 t - \left(\kappa^*\theta^*\right)^T \left(M_{1,t} - It\right)\kappa^{*-1T}\delta_1 - \frac{1}{2}\delta_1^T \kappa^{*-1}\left(\Sigma\Sigma^T t - \Sigma\Sigma^T M_{1,t}^T - M_{1,t}\Sigma\Sigma^T + M_{2,t}\right)\kappa^{*-1T}\delta_1 \tag{38}$$

Note that returning to (36), the expression for the depreciation rate over t follows

$$Q(t,T) = \frac{1}{t}\left[A(t,T) + B(t,T)^T X_t\right] \tag{39}$$

Turning to estimation in brief, a common recursive Kalman-filter-based maximum likelihood method produces the parameters (e.g., Kim and Wright, 2005). Very briefly, in state-space form, the measurement equation follows

$$\underset{q\times 1}{Y_t} = \underset{q\times 1}{\Lambda} + \underset{q\times n}{\mathrm{H}^T}\ \underset{n\times 1}{X_t} + \underset{q\times 1}{\eta_t} \tag{40}$$

where Y is a $q\times 1$ vector of observed data, namely log differences between forward quotes and spot foreign exchange rates (at q selected forward horizons); Λ is a $q\times 1$ vector; H is a $n\times q$ matrix; again the vector X_t represents the unobservable state variables; and η is a vector of Gaussian measurement errors. Given the assumed stochastic Ornstein-Uhlenbeck process in (32), the transition equation (from one discrete observation to the next) is

$$X_t = e^{-\kappa}X_{t-1} + \left(I - e^{-\kappa}\right)\theta + \omega_t \tag{41}$$

where ω is a zero-mean Gaussian error vector.

7. Empirical Results: Selected Time-Series and Cross-Sections

The sample includes 11 \$U.S. currency pairs, and as Exhibit 1 indicates, coverage ranges from the mid- to late-1990s through early 2015. True, the sample period is somewhat short, yet the length is comparable to, if not longer than, common applications to ATSMs (e.g., Dai and Singleton, 2000; Kim and Wright 2005; Kim and Orphanides, 2005), and the tradeoff between parameter stability and robustness of course arises with longer time series. Also, the frequency of Y for the Kalman filter is weekly (Wednesday), with 1-, 3-, 6-, 9-, 12-, 24-, and in cases where available 36-month forward horizons (i.e., the length of q). Even the longest horizon is perhaps short compared to ATSM calibrations, which commonly use maturities to 10 years, but besides data unavailability, the limited span is perhaps not disadvantageous given the presumed stochastic process. That is, as long as X_t is stationary following (32), anticipated depreciation must converge to δ_0 as the horizon lengthens, and calibrated distant-horizon forward quotes owe largely to the premium by construction. However, the precise juncture when expected depreciation asymptotes to δ_0 is likely beyond the 3-year horizon, and besides, the implied depreciation paths should speak to this issue.

Turning to the results, again the closed-form solution to the model(s) imply non-zero premiums under all but the most restrictive parameters, and the estimate of the discount anomaly at a given horizon is simply the difference between the AFCM implied forward (i.e., log-difference in the forward and spot exchange rates) less the corresponding AFCM-implied depreciation rate. In Exhibit 2, the dashed black lines, the solid black lines, the blue lines, and the red lines for each pair represent the observe forward quotes (in terms of log differences with the spot rate), the AFCM-implied forward, the AFCM-implied expected depreciation rate, and the AFCM-implied discount premium, respectively. Also, for reference and without regard to any test of IRP, the green lines show the corresponding spread between U.S. Treasury (UST) and the foreign government bond yield, German Bunds in the case of the euro. The results alternatively refer to 3- and 4-factor models (i.e., alternative assumptions about *n*), depending on data fit.

Three simple empirical questions for application are, first, whether the model fits the data, second, whether it indeed produces a non-zero discount premium, and third, whether the estimated premium varies over time. To address the first question, judging from the largely minuscule visual distinction between the dashed and solid black lines as well as the simple ratio of the residual to explained sum of squares (reported in Column 6 in Exhibit 1), the models seem to fit very closely. Even those pairs for which 4-factor models fit the data better, corresponding 3-factor models also produce modest errors. Regarding the second question, the red lines in Exhibit 2 indeed diverge from zero to varying degrees across pairs. Therefore, the AFCMs produce a (risk-neutral) estimate of the discount anomaly.

With respect to the third question, the AFCM-implied discount premiums indeed change over time. The degree of variability—at least for the 2- and 3-year forward horizons differs somewhat across pairs, and there is some cross-sectional variation along the forward term structure. For example, the red lines in Exhibit 3—which show the schedule of depreciation by horizon for the most recent sample date—indicate meaningful and in some cases non-linear slopes. In other words, the AFCMs produce cross-sectional as well as time-series variation in the discount premiums, which might help inform more precise assessments of forward-implied deprecation rates over a given investment horizon.

8. Empirical Results: Previous Literature on the Premium Anomaly

These results may also be noteworthy for the broader (academic) literature on the forward premium anomaly. Using regressions that include contemporaneous and realized quotes of spot and forward rates, Fama (1984) as well as Hodrick and Srivastava (1986) (FHS) and others find that the premium is more volatile than expected depreciation and that the two quantities are negatively correlated. On the other hand, although the derived premium is non-zero and variable, survey-based measures in general suggest that it is less volatile than expected depreciation, and that the correlation is generally non-negative (e.g., Froot and Frankel, 1989; Chinn and Frankel, 1994; Chinn and Frankel, 2002).

In short, Exhibit 4 suggests that the AFCM-implied premium and expected depreciation rate series are, on balance, more consistent with the survey-based results, albeit of course based in distinct underlying methods and samples. The top panel shows the simple correlation coefficients between the two series for the full daily samples at the 1-, 3-, 6-, 12-, and 24-month horizons. Although for seven of 11 cases the correlation is less than zero at the longest horizon, the figures are only consistently negative across these horizons for the NOK and SEK. Also, the lower panel shows the ratio of the variance of the premium and expected depreciation rates, which is less than one for every pair and horizon, in most cases markedly so. Therefore, the results are broadly inconsistent with FHS, as well as Bilson (1981), who argues that expected depreciation is always zero and that changes in forward rates owe exclusively to the premium. This is a very strong inference in the context of IRP. Indeed, if expected depreciation is always zero, then the anticipated spread in expected short rates, perhaps independently derived from ATSMs, must also always be zero. Forward rates would exclusively embed, in absolute contrast to the EH, term premiums and no information whatsoever about investors' expectations for the path of monetary policy. Instead, the AFCM results broadly reflect common applications of ATSMs, insofar as indeed forwards embed some useful information about expectations.

9. Caveats, Uses, and Extensions of AFCMs

No more than ATSMs validate the pure expectations theory of interest rates, the preceding closed-form and empirical analyses do not endeavor to reconcile fully the discount premium anomaly with any version of the efficient markets or rational expectations hypotheses.[4] Rather, the more limited aim is to derive models that isolate the (risk-neutral) premium and therefore anticipated depreciation rates. Returning to Exhibit 3, the estimated expected depreciation term structures—the gaps between the black and blue lines—suggest that forward quotes are indeed not unbiased estimates of expected returns. True, at least for the most recent sample date, there are many pairs for which AFCM-implied expected depreciation closely tracks log differences between either quoted or fitted forward and spot rates. But, expected deprecation meaningfully diverges from raw forward quotes in some notably cases—for example, around 70 basis points over a 2-year horizon for the EUR.

Turning to extensions, AFCMs might indeed inform market-based returns in an amended Black-Litterman (1992) framework that relaxes the "reverse-optimization" assumption, which in effect assumes index efficiency (e.g., Sharpe, 1976). But even so, obviously active management requires views that diverge from consensus, and of course the preceding only addresses investors' expectations. Differences between fitted and actual forward rates implied by AFCMs could be interpreted as valuation gaps, but the key objective of these analyses is to disentangle premiums from expected depreciation. Also, there are alternatives to a pure Gaussian process for

[4] For example, under the condition of pro-cyclical risk-free rates, Verdelhan (2010) outlines a two-country model with external habit preferences that replicates the forward premium puzzle.

the underlying factors, including, say, jump-diffusion, which might better capture exchange rate movements. Nevertheless, the Ornstein-Uhlenbeck process in (32), which in the context of models such as Vasicek (1977) problematically allows negative nominal interest rates, is arguably better suited for currencies, which can of course depreciation or appreciate.

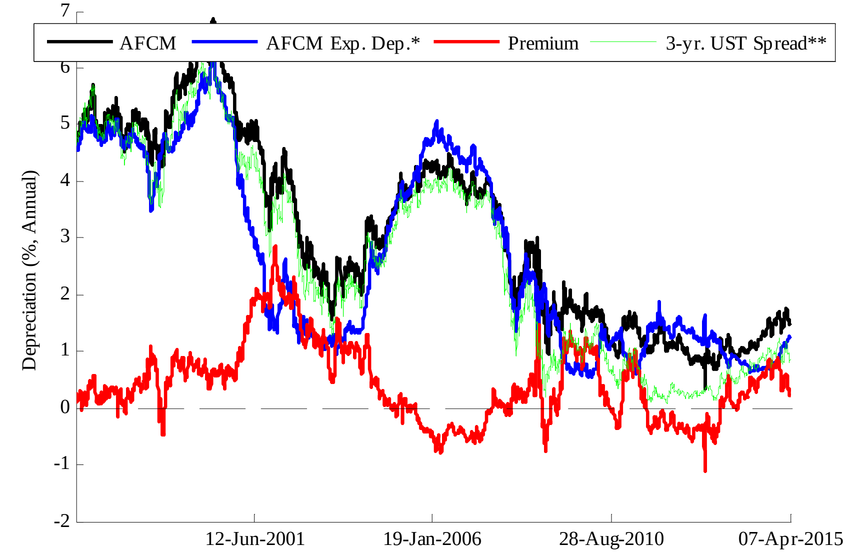
Exhibit 1: Expected $U.S. Depreciation Rates

Expected $U.S. Depreciation
4/7/2015

		-1-	-2-	-3-	-4-	-5-	-6-	-7-	-8-	-9-
	Horizon:	1-Month	3-Month	6-Month	12-Month	24-Month	1-ESS/RSS	Factors	Sample Start	Sample End
Japanese Yen	JPY	0.46%	0.52%	0.61%	0.77%	1.04%	0.99683	3	11/04/96	04/07/15
European Union Euro	EUR	0.49%	0.50%	0.43%	0.32%	0.28%	0.99779	4	10/07/99	04/07/15
U.K. Pound Sterling	GBP	-0.28%	-0.38%	-0.45%	-0.55%	-0.69%	0.99591	4	11/28/96	04/07/15
Canadian Dollar	CAD	-0.52%	-0.52%	-0.45%	-0.34%	-0.29%	0.98889	3	04/30/98	04/07/15
Australian Dollar	AUD	-2.00%	-1.76%	-1.53%	-1.19%	-0.70%	0.98988	3	01/06/98	04/07/15
Swiss Franc	CHF	1.51%	1.63%	1.73%	1.81%	1.86%	0.99436	3	10/07/99	04/07/15
Swedish Krona	SEK	0.49%	0.52%	0.52%	0.48%	0.43%	0.99771	4	02/28/96	04/07/15
Norwegian Krone	NOK	-1.10%	-0.74%	-0.27%	0.64%	2.50%	0.99851	4	12/08/97	04/07/15
New Zealand Dollar	NZD	-3.52%	-3.54%	-3.49%	-3.36%	-3.14%	0.99378	3	11/15/96	04/07/15
Danish Krone	DKK	1.60%	1.52%	1.33%	1.07%	0.85%	0.99685	4	07/15/99	04/07/15
Singapore Dollar	SGD	-0.91%	-1.05%	-1.05%	-0.94%	-0.73%	0.9979	4	08/11/99	04/07/15

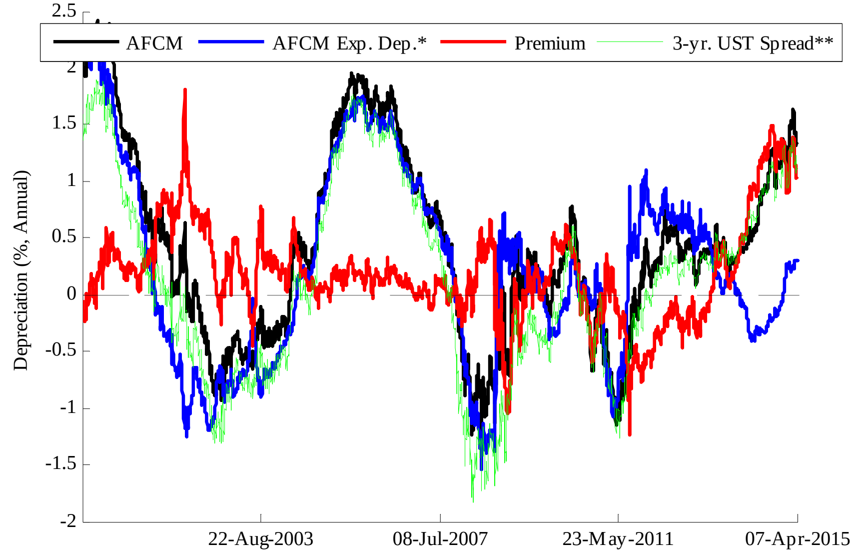
Exhibit 2: Time Series Results

$US/JPY:3-factor Affine Forward Currency Model (AFCM) (3-year Horizon)
11/04/1996-04/07/2015



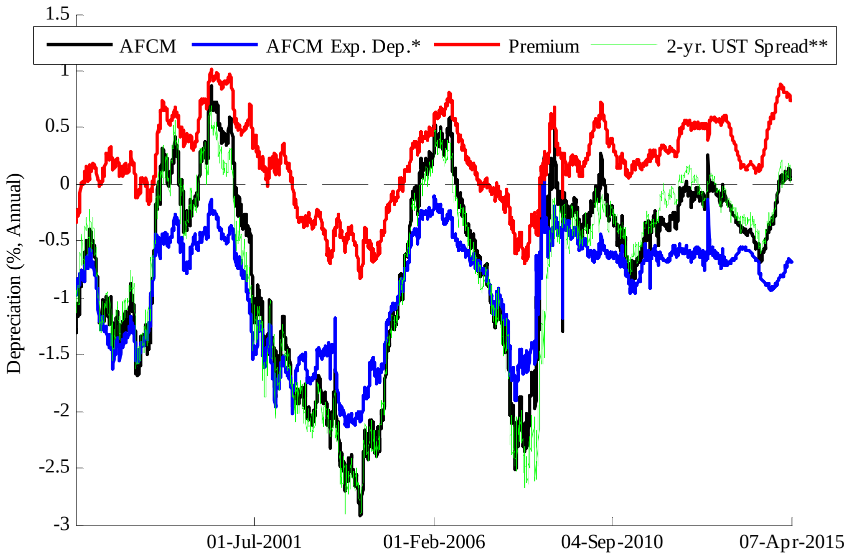
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (3-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.91742 (0.19431).

$US/EUR:4-factor Affine Forward Currency Model (AFCM) (3-year Horizon)
10/07/1999-04/07/2015



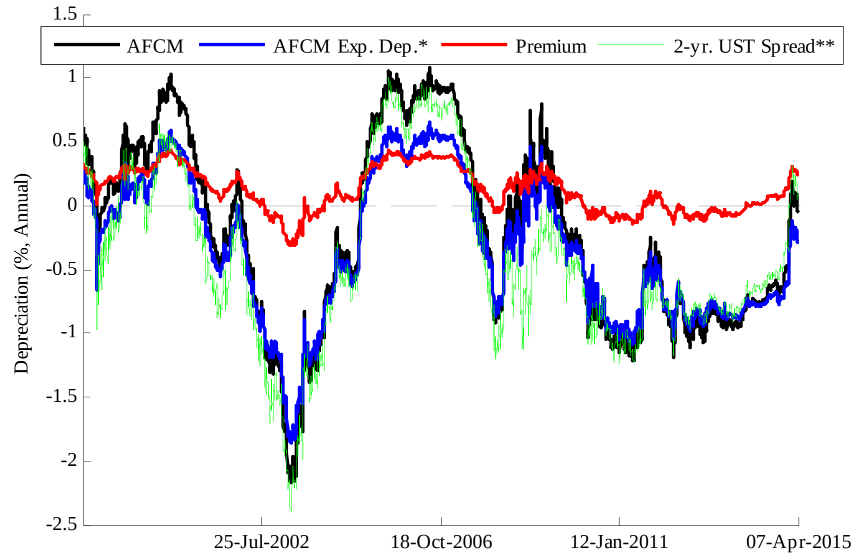
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (3-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.86532 (0.2198).

$US/GBP:4-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
11/28/1996-04/07/2015



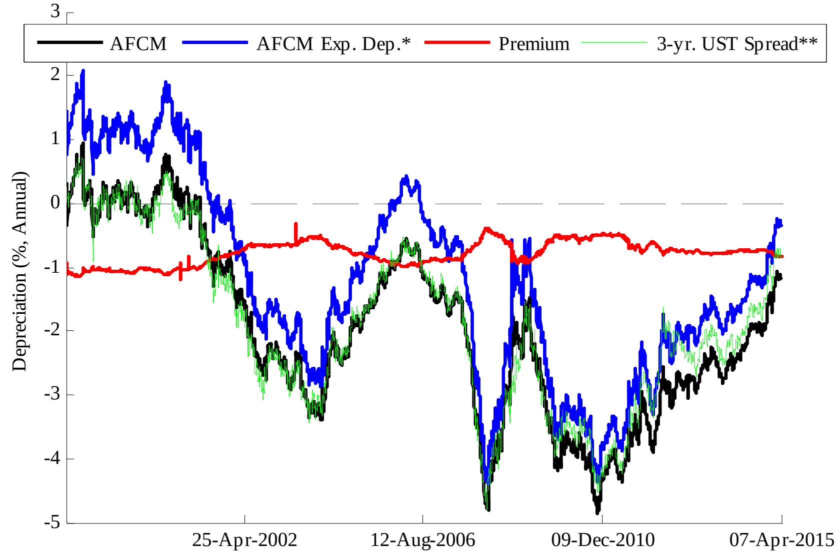
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.95418 (0.9236).

$US/CAD:3-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
04/30/1998-04/07/2015



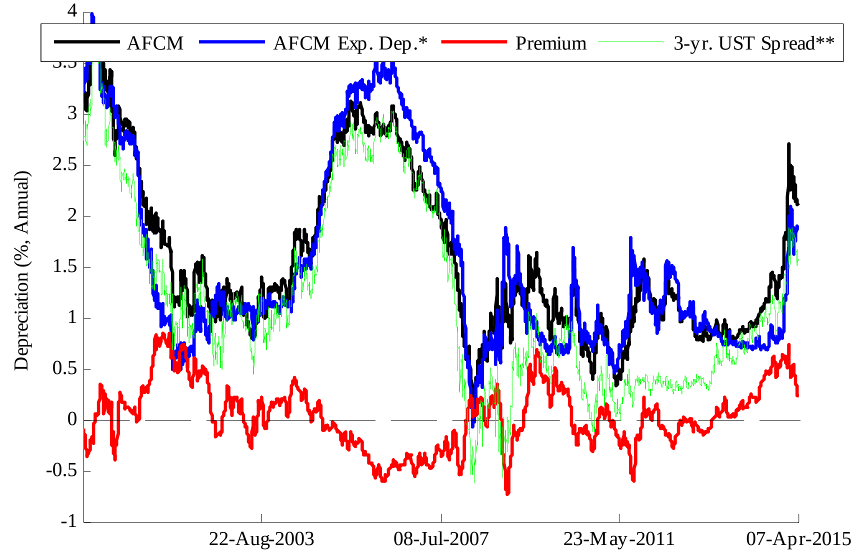
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.99883 (0.98744).

$US/AUD:3-factor Affine Forward Currency Model (AFCM) (3-year Horizon)
01/06/1998-04/07/2015



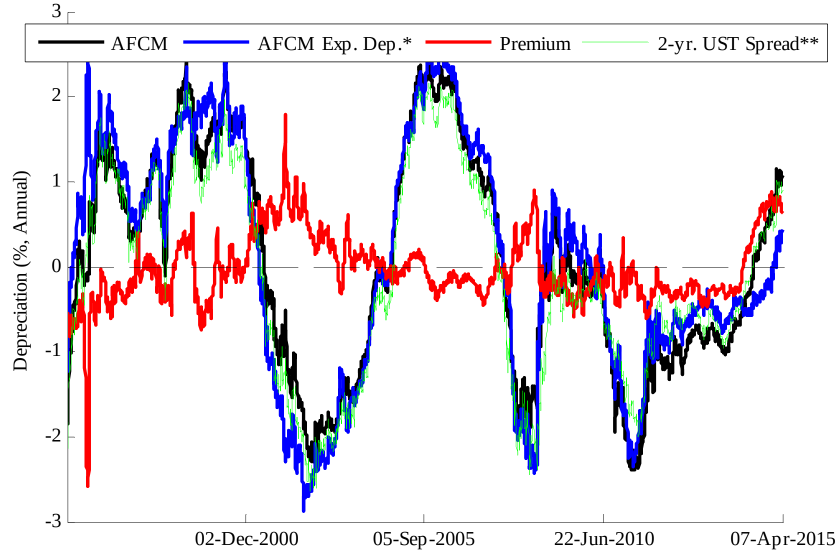
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (3-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.99935 (-0.95145).

$US/CHF:3-factor Affine Forward Currency Model (AFCM) (3-year Horizon)
10/07/1999-04/07/2015



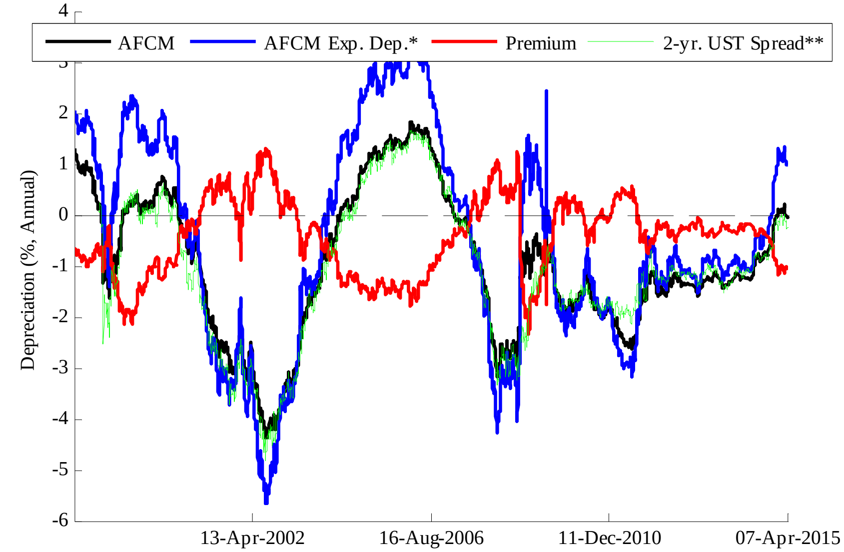
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (3-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.94541 (-0.22096).

$US/SEK:4-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
02/28/1996-04/07/2015



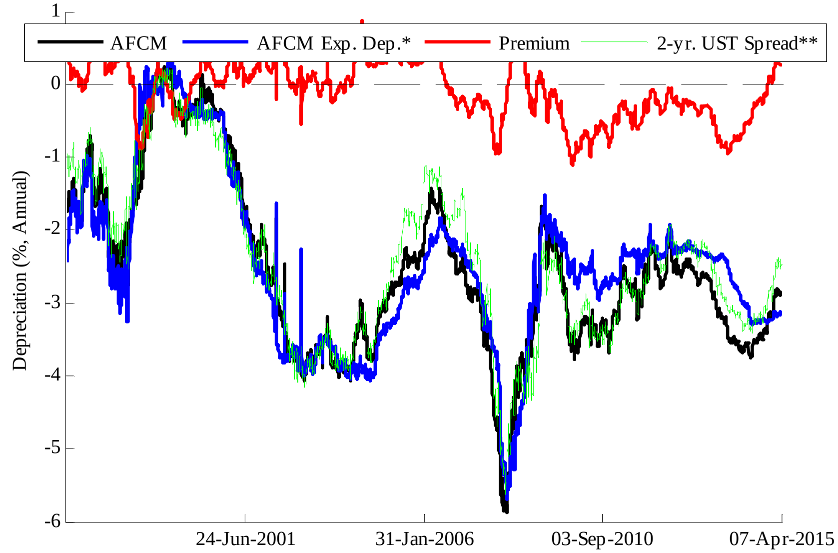
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.95693 (-0.12076).

$US/NOK:3-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
12/08/1997-04/07/2015



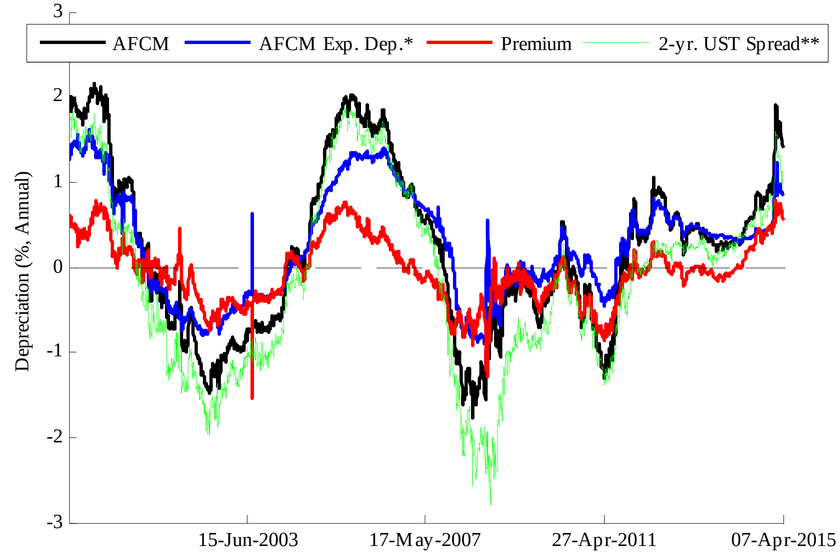
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.98015 (-0.8325).

$US/NZD:3-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
11/15/1996-04/07/2015



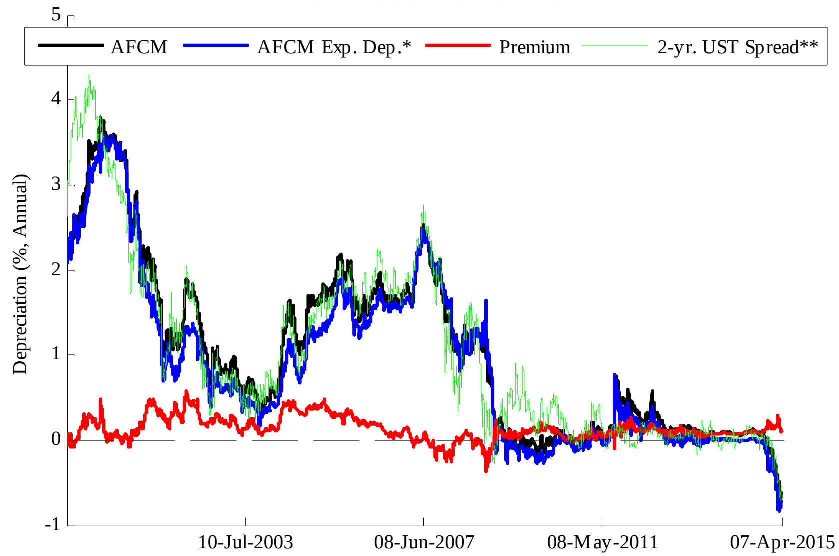
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.93983 (0.22713).

$US/DKK:4-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
07/15/1999-04/07/2015



Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.97378 (0.92583).

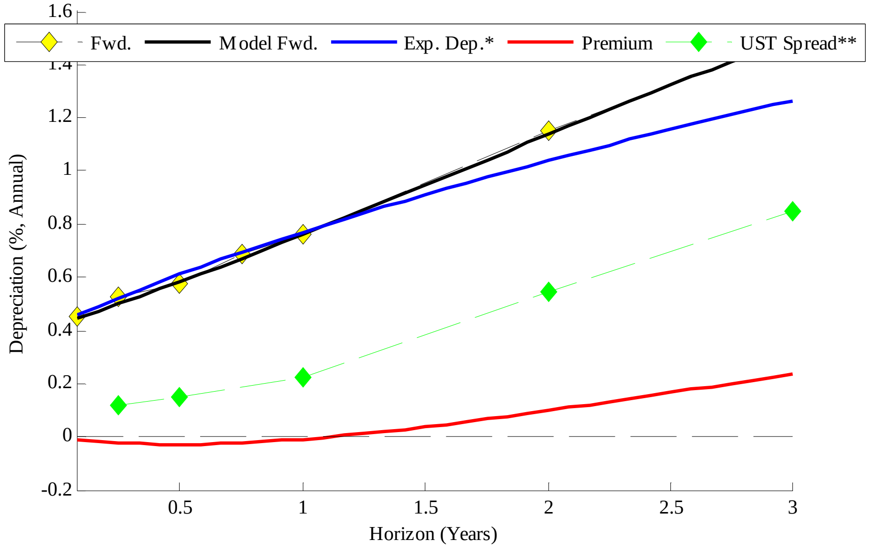
$US/SGD:4-factor Affine Forward Currency Model (AFCM) (2-year Horizon)
08/11/1999-04/07/2015



Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield
Sample Correlation between (2-yr) AFCM Fwd. and AFCM Exp. Dep (AFCM Premium) =0.99077 (0.17576).

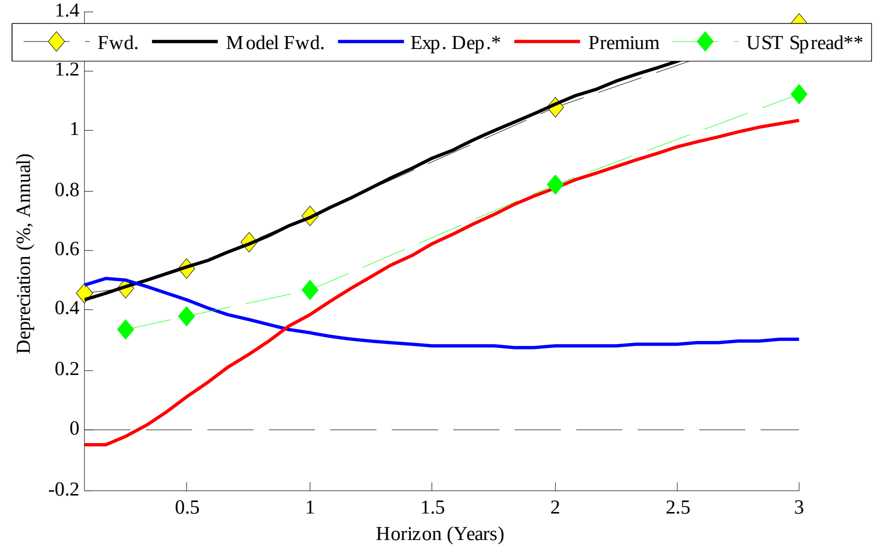
Exhibit 3: Term Structures of Expected Depreciation

$US/JPY:3-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-15.879%)
04/07/2015



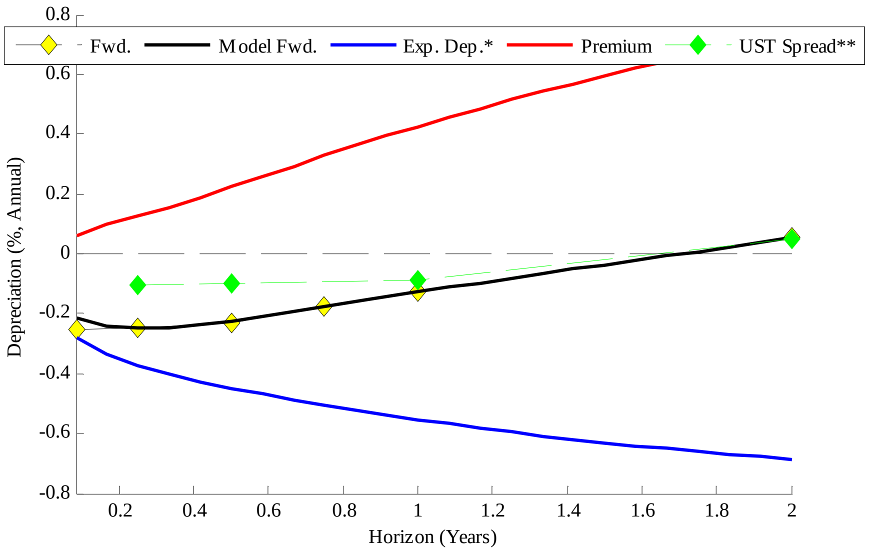
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/EUR:4-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-24.332%)
04/07/2015



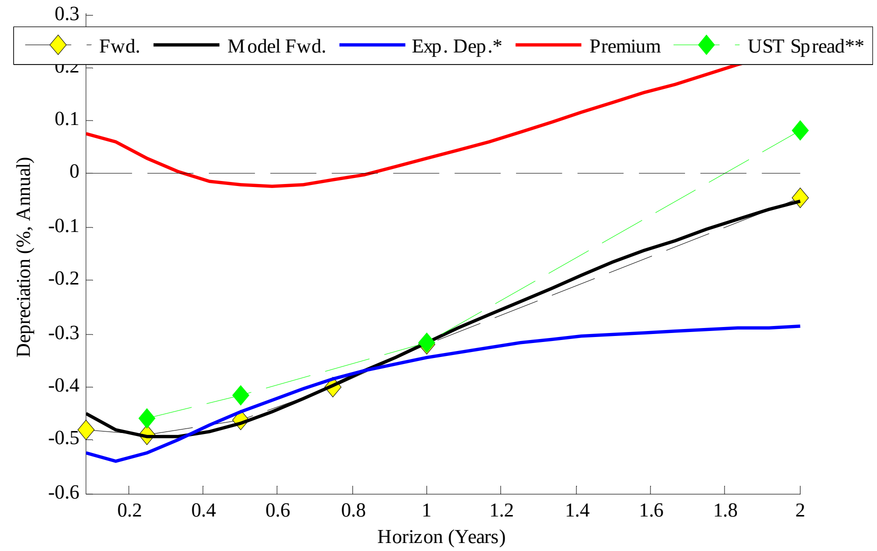
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/GBP:4-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-12.5735%)
04/07/2015



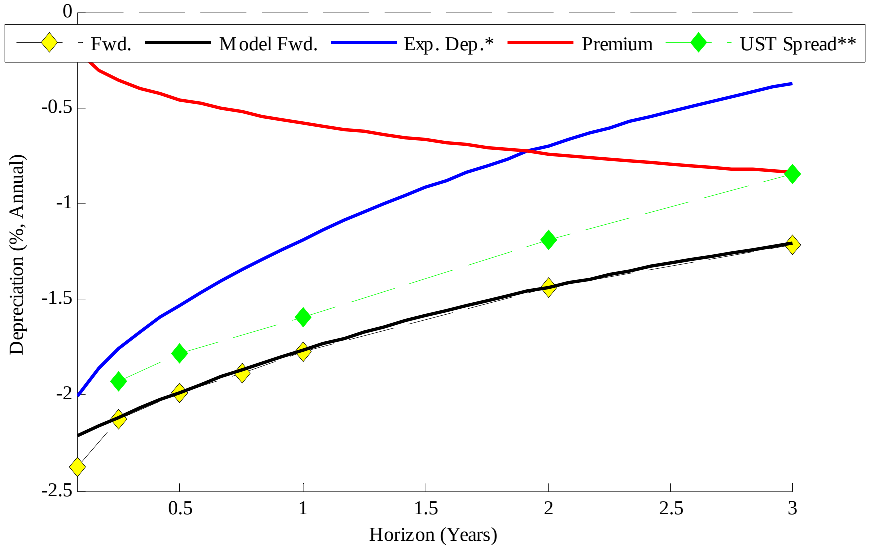
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/CAD:3-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-12.7405%)
04/07/2015



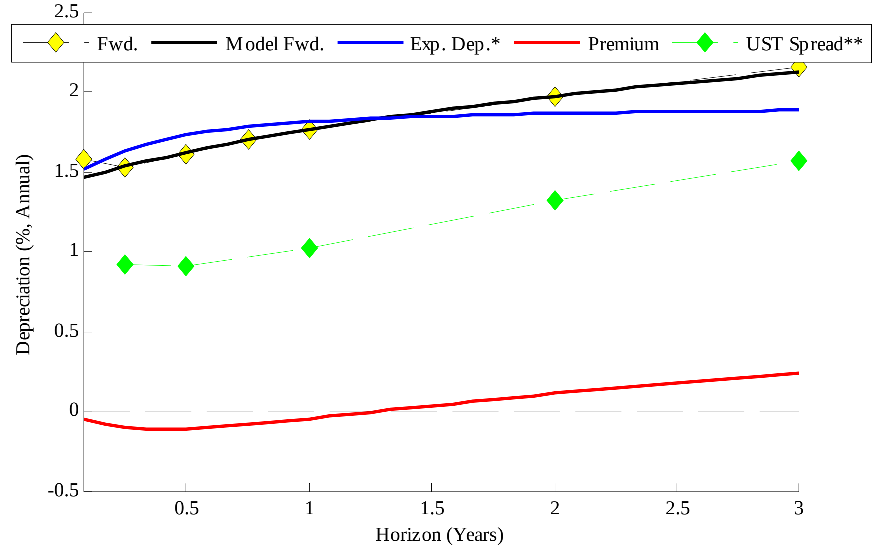
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/AUD:3-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-20.054%)
04/07/2015



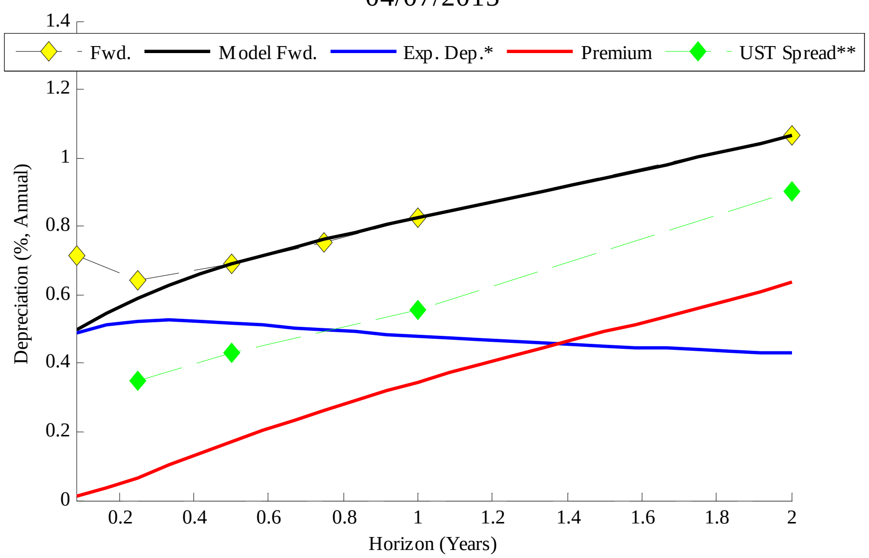
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/CHF:3-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-8.7631%)
04/07/2015



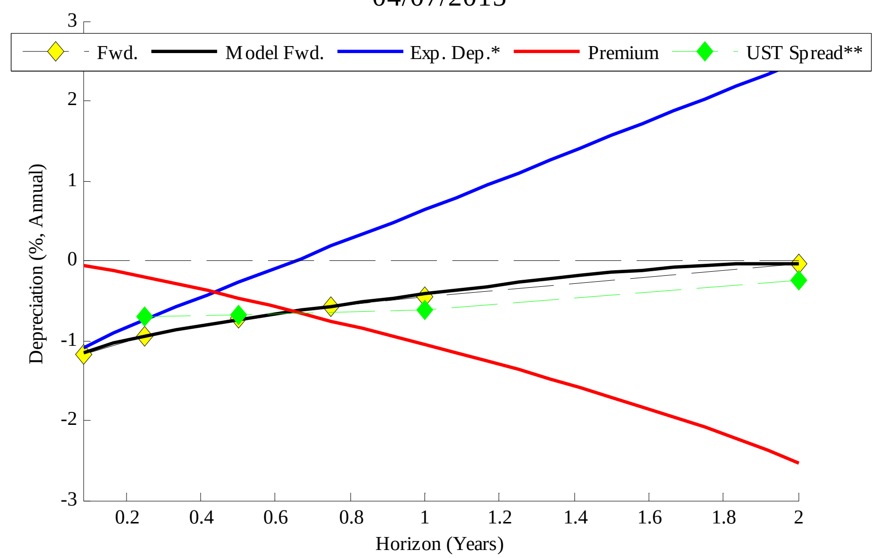
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/SEK:4-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-27.1814%)
04/07/2015



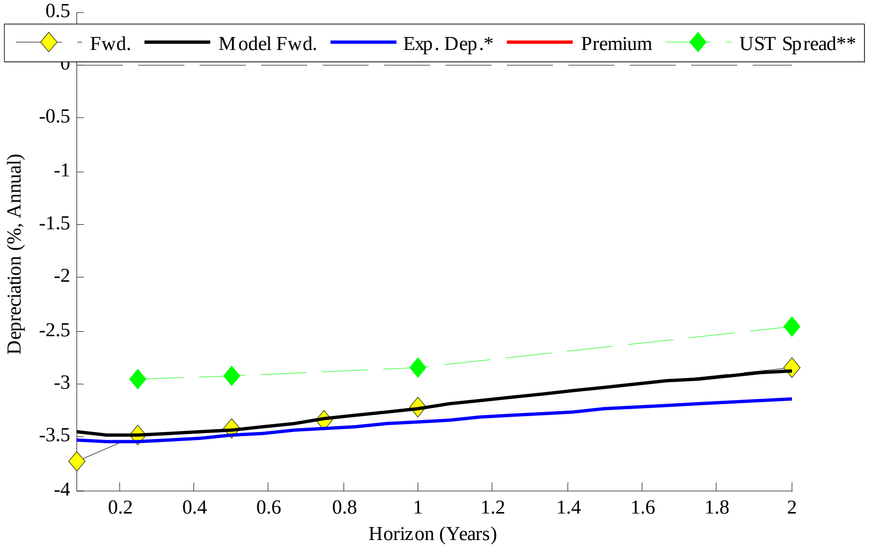
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/NOK:4-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-29.8231%)
04/07/2015



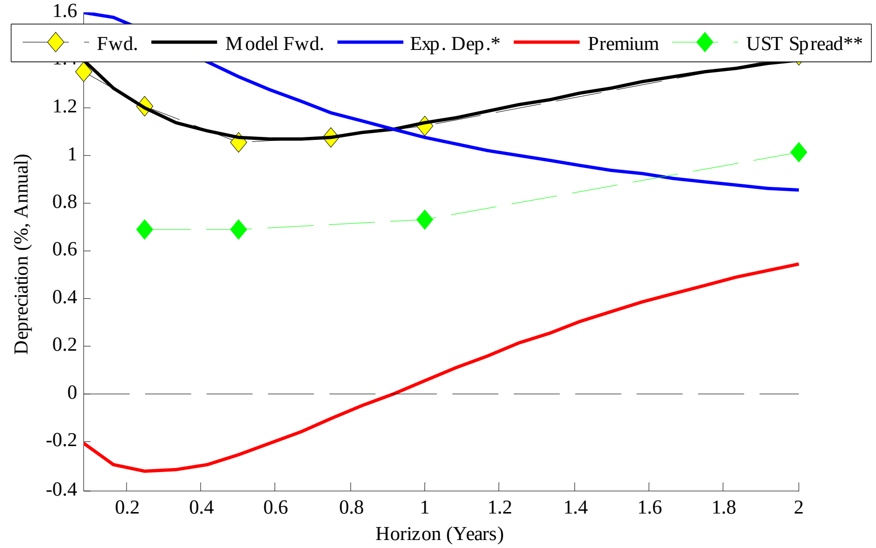
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/NZD:3-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/18/2014-04/07/2015) = (-13.5564%)
04/07/2015



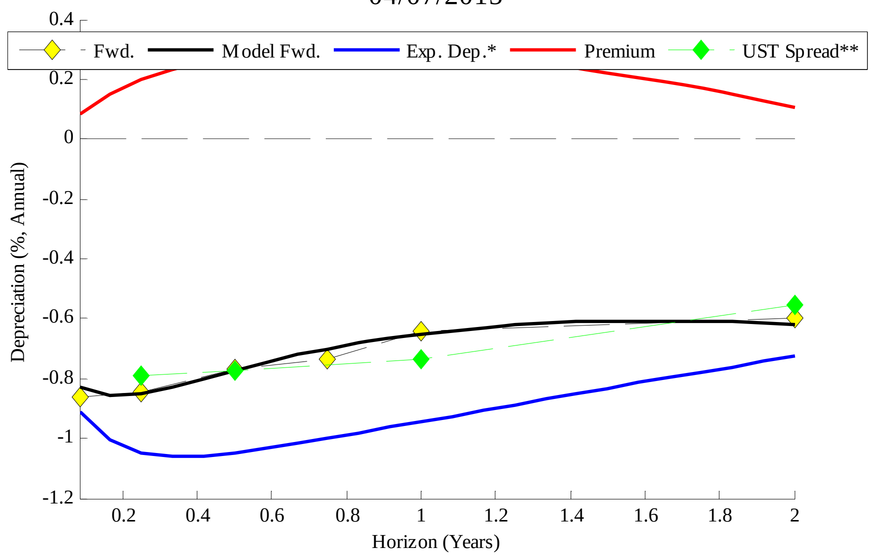
Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/DKK:4-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-24.4003%)
04/07/2015



Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

$US/SGD:4-factor Affine Forward Currency Model (AFCM) Term Structure
Realized Depreciation (04/21/2014-04/07/2015) = (-8.2465%)
04/07/2015



Sources: Author's calculations, Bloomberg. *Expressed as the $US price of a single foreign currency unit.
**US Treasury (UST) zero-coupon par spread over foreign government bond yield

Exhibit 4:

Fama (1984) Conditions
Correlation Coefficients:
AFCM Expected Depreciation & Premium

		-1-	-2-	-3-	-4-	-5-	-6-	-7-
	Horizon:	1-Month	3-Month	6-Month	12-Month	24-Month	Sample Start	Sample End
Japanese Yen	JPY	0.11298	0.089003	0.073681	0.017982	-0.11146	11/04/96	04/07/15
European Union Euro	EUR	0.10469	0.16471	0.14635	-0.10129	-0.25589	10/07/99	04/07/15
U.K. Pound Sterling	GBP	0.31619	0.65911	0.80366	0.80002	0.76656	11/28/96	04/07/15
Canadian Dollar	CAD	0.49749	0.58101	0.81928	0.97684	0.97865	04/30/98	04/07/15
Australian Dollar	AUD	0.69401	0.72148	0.75517	0.6642	-0.54425	01/06/98	04/07/15
Swiss Franc	CHF	0.18719	0.053616	-0.18475	-0.52695	-0.48411	10/07/99	04/07/15
Swedish Krona	SEK	-0.40889	-0.49251	-0.39112	-0.34647	-0.40374	02/28/96	04/07/15
Norwegian Krone	NOK	-0.15988	-0.1164	-0.25052	-0.58887	-0.89968	12/08/97	04/07/15
New Zealand Dollar	NZD	0.0078596	-0.20822	-0.29344	-0.081222	-0.11924	11/15/96	04/07/15
Danish Krone	DKK	0.32183	0.44221	0.65271	0.84477	0.81558	07/15/99	04/07/15
Singapore Dollar	SGD	0.13905	-0.12847	-0.29487	-0.32549	0.040668	08/11/99	04/07/15

Variance Ratios:
AFCM Expected Depreciation & Premium

Japanese Yen	JPY	0.00030535	0.0015288	0.0034415	0.013315	0.07543	11/04/96	04/07/15
European Union Euro	EUR	0.0012922	0.0012662	0.0019321	0.023545	0.12414	10/07/99	04/07/15
U.K. Pound Sterling	GBP	0.0081798	0.012592	0.034413	0.1497	0.60928	11/28/96	04/07/15
Canadian Dollar	CAD	0.035985	0.051763	0.045194	0.056843	0.093581	04/30/98	04/07/15
Australian Dollar	AUD	0.019986	0.037612	0.03332	0.019015	0.0011993	01/06/98	04/07/15
Swiss Franc	CHF	0.00071097	0.0023065	0.0019476	0.004107	0.043492	10/07/99	04/07/15
Swedish Krona	SEK	0.00027129	0.0008516	0.0031008	0.015903	0.085525	02/28/96	04/07/15
Norwegian Krone	NOK	0.00091571	0.012162	0.064539	0.2684	0.70219	12/08/97	04/07/15
New Zealand Dollar	NZD	0.011108	0.006187	0.0030936	0.028358	0.12307	11/15/96	04/07/15
Danish Krone	DKK	0.0056491	0.024353	0.050885	0.11787	0.36232	07/15/99	04/07/15
Singapore Dollar	SGD	0.0011069	0.0053779	0.011237	0.01441	0.018966	08/11/99	04/07/15

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Appendix A: Solution to a multi-factor Gaussian AFCM

To solve the PDE, (35), for the multi-factor model, note that the relevant partial derivatives of the proposed affine solution, (36), follow

$$\begin{aligned} \frac{\partial F}{\partial X} &= FB \\ \frac{\partial^2 F}{\partial X \partial X^T} &= FBB^T \\ \frac{\partial F}{\partial t} &= F\left(\frac{\partial A(t,T)}{\partial t} + \frac{\partial B(t,T)}{\partial t}^T X \right) \end{aligned} \tag{A.42}$$

With substitution and the fact that *F* is a scalar, the PDE reduces to the following two ODEs,

$$\begin{aligned} &\frac{\partial A(t,T)}{\partial t} + \frac{\partial B(t,T)}{\partial t}^T X + B^T \kappa(\theta - X_t) - B^T \Sigma(\lambda_0 + \lambda_1^T X_t) + \frac{1}{2} B^T \Sigma \Sigma^T B - \delta_0 - \delta_1^T X_t = 0 \\ &\frac{\partial A(t,T)}{\partial t} + B^T \kappa\theta - B^T \Sigma \lambda_0 + \frac{1}{2} B^T \Sigma\Sigma^T B - \delta_0 = \left(-\frac{\partial B(t,T)}{\partial t}^T + B^T \kappa + B^T \Sigma \lambda_1^T + \delta_1^T \right) X \\ &\frac{\partial A(t,T)}{\partial t} + B^T (\kappa\theta - \Sigma\lambda_0) + \frac{1}{2} B^T \Sigma\Sigma^T B - \delta_0 = \left[-\frac{\partial B(t,T)}{\partial t}^T + B^T (\kappa + \Sigma\lambda_1^T) + \delta_1^T \right] X \end{aligned} \tag{A.43}$$

Analogous to the single-factor model, the right-hand-side, with $\kappa^* = \kappa + \Sigma\lambda_1$, must follow (after taking the transpose)

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t}^T &= B^T \kappa^* + \delta_1^T \\ \frac{\partial B(t,T)}{\partial t} &= \kappa^{*T} B + \delta_1 \end{aligned} \tag{A.44}$$

To solve (A.44), use the integrating factor, take the definite integral, and use the initial condition following

$$\exp\left[\kappa^{*T}t\right]\left[\frac{\partial B(t,T)}{\partial t}-\kappa^{*T}B\right]=\exp\left[\kappa^{*T}t\right]\delta_1$$

$$\int_0^t \frac{\partial}{\partial s}\left[\exp\left[\kappa^{*T}t\right]B(t,T)\right]ds=\int_0^t \exp\left[\kappa^{*T}s\right]\delta_1 ds$$

$$\exp\left[\kappa^{*T}t\right]B(t,T)-\exp\left[\kappa^{*T}t\right]B(0,T)=\left(\exp\left[\kappa^{*T}s\right]-I\right)\left(\kappa^{*T}\right)^{-1}\Big|_0^t\,\delta_1 \tag{A.45}$$

$$\exp\left[\kappa^{*T}t\right]B(t,T)=\left(\exp\left[\kappa^{*T}t\right]-I\right)\left(\kappa^{*T}\right)^{-1}\delta_1$$

$$B(t,T)=\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[\kappa^{*T}t\right]\right)\delta_1$$

Given (A.45); with $\kappa^{*-1}\left(\kappa\theta_t-\Sigma\lambda_0\right)=\theta^*$, which implies $\kappa\theta-\Sigma\lambda_0=\kappa^*\theta^*$; and the fact that since $B(t,T)$ and $\kappa^*\theta^*$ are $n\times 1$, then $B(t,T)^T\left(\kappa^*\theta^*\right)=\left(\kappa^*\theta^*\right)^T B(t,T)$; the second ODE becomes

$$\frac{\partial A(t,T)}{\partial t}=\delta_0-\left(\kappa^*\theta^*\right)^T\left[\left(\kappa^{*T}\right)^{-1}\left(I-e^{-\kappa^{*T}t}\right)\delta_1\right]-\frac{1}{2}\left[\left(\kappa^{*T}\right)^{-1}\left(I-e^{-\kappa^{*T}t}\right)\delta_1\right]^T\Sigma\Sigma^T\left[\left(\kappa^{*T}\right)^{-1}\left(I-e^{-\kappa^{*T}t}\right)\delta_1\right] \tag{A.46}$$

Taking one of the transposes in the third term on the right-hand-side and subsequent matrix multiplication follows

$$\begin{aligned}
\frac{\partial A(t,T)}{\partial t}&=\delta_0-\left(\kappa^*\theta^*\right)^T\left[\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[-\kappa^{*T}t\right]\right)\delta_1\right]-\frac{1}{2}\delta_1^T\left(I-\exp\left[-\kappa^{*T}t\right]\right)\kappa^{*-1}\Sigma\Sigma^T\left[\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[-\kappa^{*T}t\right]\right)\delta_1\right]\\
&=\delta_0-\left(\kappa^*\theta^*\right)^T\left[\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[-\kappa^{*T}t\right]\right)\delta_1\right]-\frac{1}{2}\delta_1^T\kappa^{*-1}\left(I-\exp\left[-\kappa^{*T}t\right]\right)\Sigma\Sigma^T\left(I-\exp\left[-\kappa^{*T}t\right]\right)\left(\kappa^{*T}\right)^{-1}\delta_1\\
&=\delta_0-\left(\kappa^*\theta^*\right)^T\left[\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[-\kappa^{*T}t\right]\right)\delta_1\right]-\frac{1}{2}\delta_1^T\kappa^{*-1}\left(\Sigma\Sigma^T-\Sigma\Sigma^T\exp\left[-\kappa^{*T}t\right]-\exp\left[-\kappa^{*T}t\right]\Sigma\Sigma^T+\exp\left[-\kappa^{*T}t\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}t\right]\right)\left(\kappa^{*T}\right)^{-1}\delta_1
\end{aligned}$$

(A.47)

Definite integrals produce the solution to (A.47). Given the initial condition, the left-hand-side becomes

$$\int_0^t dA(s,T)=A(t,T)-A(0,T)=A(t,T) \tag{A.48}$$

For the second term on the right-hand-side of (A.47), let $M_{1,t}=\left(\kappa^{*T}\right)^{-1}\left(I-e^{-\kappa^{*}Tt}\right)$, and the integration follows

$$
\begin{aligned}
\int_0^t \left(\kappa^*\theta^*\right)^T \left[\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[-\kappa^{*T}s\right]\right)\delta_1\right]ds &= \left(\kappa^*\theta^*\right)^T \left[\int_0^t \left(I-\exp\left[-\kappa^{*T}s\right]\right)ds\right]\left(\kappa^{*T}\right)^{-1}\delta_1 \\
&= \left(\kappa^*\theta^*\right)^T \left\{It-\left[-\left(\kappa^{*T}\right)^{-1}\left(\exp\left[-\kappa^{*T}s\right]\Big|_0^t\right)\right]\right\}\left(\kappa^{*T}\right)^{-1}\delta_1 \\
&= \left(\kappa^*\theta^*\right)^T \left[It+\left(\kappa^{*T}\right)^{-1}\left(\exp\left[-\kappa^{*T}s\right]-I\right)\right]\left(\kappa^{*T}\right)^{-1}\delta_1 \qquad \text{(A.49)} \\
&= \left(\kappa^*\theta^*\right)^T \left[It-\left(\kappa^{*T}\right)^{-1}\left(I-\exp\left[-\kappa^{*T}s\right]\right)\right]\left(\kappa^{*T}\right)^{-1}\delta_1 \\
&= \left(\kappa^*\theta^*\right)^T \left[It-M_{1,t}\right]\left(\kappa^{*T}\right)^{-1}\delta_1
\end{aligned}
$$

For the third term on the right-hand-side, we need to calculate three (matrix) integrals, including

$\int_0^t \exp\left[-\kappa^{*\mathrm{T}}s\right]ds$, $\int_0^t \exp\left[-\kappa^{*}s\right]ds$, and $\int_0^t \exp\left[-\kappa^{*\mathrm{T}}s\right]\Sigma\Sigma^T \exp\left[-\kappa^{*\mathrm{T}}s\right]ds$. The first follows

$$
\int_0^t \exp\left[-\kappa^{*\mathrm{T}}s\right]ds = -\kappa^{*-1T}\exp\left[-\kappa^{*\mathrm{T}}s\right]\Big|_0^t = \kappa^{*-1T}\left(I-\exp\left[-\kappa^{*\mathrm{T}}t\right]\right) = M_{1,t} \qquad \text{(A.50)}
$$

Similarly, the second follows

$$
\int_0^t \exp\left[-\kappa^{*}s\right]ds = \kappa^{*-1}\left(I-\int_0^t \exp\left[-\kappa^{*}t\right]\right) = M_{1,t}^T \qquad \text{(A.51)}
$$

For the third integral, first use the product rule and take the integrals, as in

$$
\begin{aligned}
&\frac{d}{ds}\left(\exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]\right) = -\kappa^{*}\exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right] + \exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]\left(-\kappa^{*T}\right) \\
&\int_0^t d\left(\exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]\right) = \int_0^t -\kappa^{*}\exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]ds - \int_0^t \exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]\kappa^{*T}ds \\
&\exp\left[-\kappa^{*}t\right]\Sigma\Sigma^T e^{-\kappa^{*T}t} - \Sigma\Sigma^T = -\kappa^{*}\left(\int_0^t \exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]ds\right) - \left(\int_0^t \exp\left[-\kappa^{*}s\right]\Sigma\Sigma^T\exp\left[-\kappa^{*T}s\right]ds\right)\kappa^{*T} \\
&\qquad = -\kappa^{*}\Omega_t - \Omega_t\kappa^{*T}
\end{aligned}
$$

(A.52)

with $\Omega_t = \int_0^t \exp\left[-\kappa^* s\right]\Sigma\Sigma^T \exp\left[-\kappa^{*\mathrm{T}} s\right] ds$. Vectorize both sides of the equation[5] and solve

following

$$
\begin{aligned}
-vec\left(\exp\left[-\kappa^* t\right]\Sigma\Sigma^T \exp\left[-\kappa^{*\mathrm{T}} t\right]-\Sigma\Sigma^T\right) &= vec\left(\kappa^*\Omega_t+\Omega_t\kappa^{*T}\right)\\
&= vec\left(\kappa^*\Omega_t I\right)+vec\left(I\Omega_t\kappa^{*T}\right)\\
&= \left(I\otimes\kappa^*\right)vec\left(\Omega_t\right)+\left(\kappa^*\otimes I\right)vec\left(\Omega_t\right)\\
&= \left(I\otimes\kappa^*+\kappa^*\otimes I\right)vec\left(\Omega_t\right)\\
vec\left(\Omega_t\right) &= -vec\left(\exp\left[-\kappa^* t\right]\Sigma\Sigma^T \exp\left[-\kappa^{*\mathrm{T}} t\right]-\Sigma\Sigma^T\right)\left(I\otimes\kappa^*+\kappa^*\otimes I\right)^{-1}\\
\Omega_t &= -vec^{-1}\left[vec\left(\exp\left[-\kappa^* t\right]\Sigma\Sigma^T \exp\left[-\kappa^{*\mathrm{T}} t\right]-\Sigma\Sigma^T\right)\left(I\otimes\kappa^*+\kappa^*\otimes I\right)^{-1}\right]
\end{aligned}
$$

$$
\int_0^t \exp\left[-\kappa^* s\right]\Sigma\Sigma^T \exp\left[-\kappa^{*\mathrm{T}} t\right] ds = M_{2,t} \tag{A.53}
$$

Therefore, the solution to the second ODE follows (38).

[5] Note that $vec(ABC) = \left(C^T \otimes A\right) vec(B)$.