Leverage, Moral Hazard and Liquidity[1]

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Abstract

We consider a moral hazard setup wherein leveraged firms have incentives to take on excessive risks and are thus rationed when they attempt to roll over debt. Firms can sell assets to alleviate rationing. Liquidated assets are purchased by non-rationed firms but their borrowing capacity is also limited by the risk-taking moral hazard. The market-clearing price exhibits cash-in-the-market pricing and depends on the entire distribution of leverage (debt to be rolled over) in the economy. This distribution of leverage, and its form as roll-over debt, are derived as endogenous outcomes with each firm's choice of leverage affecting the difficulty of other firms in rolling over debt in future. The model provides an agency-theoretic linkage between market liquidity and funding liquidity and formalizes the de-leveraging of financial institutions observed during crises. It also explains the role played by system-wide leverage in generating deep discounts in prices when adverse asset-quality shocks materialize in good times.

Keywords: risk-shifting, credit rationing, market liquidity, funding liquidity, fire sales, financial crises, cash-in-the-market pricing.

JEL Classification: G12, G20, D45, D52, D53

1 Introduction

"Where did all the liquidity go? Six months ago, everybody was talking about boundless global liquidity supporting risky assets, driving risk premiums to virtually nothing, and now everybody is talking about a global liquidity crunch, driving risk premiums half the distance to the moon. Tell me, Mac, where did all the liquidity go?" - Paul McCulley, PIMCO Investment Outlook, Summer 2007

Starting August 9 2007, the sub-prime crisis truly took hold of the financial sector. Since the beginning of 2007, information about the deteriorating quality of mortgage assets hit markets on a repeated basis.The impending losses for banks, broker-dealers and hedge funds involved in mortgage-backed assets, epitomized by the suspension of mark-to-market accounting by BNP Paribas' hedge funds on August 9, cast a doubt over the solvency of institutional balance-sheets. An important piece that contributed to the sharp reaction of markets was the highly short-term nature of debt with which these assets, and more broadly balance-sheets, had been financed. In particular, debt was of the asset-backed or unsecured commercial paper type that had to be rolled over at short maturities, typically one month or three months. It became progressively clear that such rollovers would be difficult given that there was substantial liquidation risk. In case assets had to be liquidated, prices would be a far cry from their "fair" or "normal-time" valuations since natural buyers of such assets were themselves hit by the shock to asset quality. Essentially, de-leveraging of the financial sector was on and this featured inability to roll over existing debt, fire sales of assets, and concerns about the ability of financial firms to meet their liabilities.

One explanation proposed as the genesis of this severe shock to asset prices is that preceding this period was a secular downward shift in macroeconomic volatility, the so-called "Great Moderation". As per this explanation, improvements in risk-sharing within and across economies were believed to have stabilized macroeconomic output. Thus credit risk of various assets was deemed to have experienced a fundamental downward revision, enabling issuance of cheaper debt than before and encouraging the build-up of leverage in the financial system. Another explanation, and these two explanations do not span the entire set, was that there was in fact a credit "bubble" fueled by short-term, risk-taking incentives of bankers and to an extent by their gaming of regulatory subsidies and prudential capital requirements.

This paper does not attempt to resolve which of these explanations is the more plausible one for the ongoing crisis of 2007-2009. Instead, its goal is to embed the risk-taking incentives of financial institutions in a (financial) industry equilibrium model where short-term rollover debt is an optimal form of financing. In particular, the model illustrates the important role played by *economy-wide* leverage in inducing asset fire sales and de-leveraging, and the role played by volatility of fundamentals in determining the economy-wide leverage in the first place. As its main deliverables, the model provides an agency-theoretic explanation for salient features of financial crises such as (i) the linkage between market liquidity and funding liquidity (put simply the ease of trading assets and raising external finance, respectively), and (ii) deep discounts observed in prices when adverse asset-quality shocks materialize in good times.

Since the backdrop we have in mind is one of trading-based financial institutions which are typically highly levered, we focus on the agency problem of asset substitution or risk-shifting by borrowers (Jensen and Meckling, 1976) wherein a borrower, after raising debt, has incentives to transfer wealth away from lenders by switching to riskier assets. Related to the work of Stiglitz and Weiss (1981) and Diamond (1989, 1991), this risk-shifting problem rations potential borrowers in that it limits the maximum amount of financing they can raise from lenders. In this setting, we show that asset sales provide a mechanism through which borrowers de-lever and relax the extent of their rationing. This simple set-up forms the building block of our model.

To analyze asset-pricing implications, we cast this building block in an industry equilibrium. Specifically, there is a continuum of financial firms which have undertaken some ex-ante debt financing (exogenous initially in the paper, endogenized later). These liabilities need to repaid or rolled over. To this end, firms attempt to raise additional debt financing, but its extent is limited due to the risk-shifting problem. The worse the asset-quality shock (for instance, interim information about asset's future prospects), the more severe is the risk-shifting problem faced by lenders, and, in turn, greater is the rationing of borrowers. Firms rationed by this debt rollover problem attempt to relax the financing constraint by liquidating some or all of their assets. These liquidated assets, however, can only be acquired by the set of remaining financial firms that has spare debt capacity (as in Shleifer and Vishny, 1992). These firms can also pledge the assets that they buy. However, they also face the moral hazard problem from ownership of risky assets, which limits their financing for asset purchase. Thus, the liquidation price, which is determined by the market-clearing condition, is of the "cash-in-the-market" type (a term introduced by Allen and Gale, 1994): When a large number of firms are liquidating assets, market price is below the expected discounted cash flow and is determined by the distribution of liquidity in the economy.

Crucially, the entire industry equilibrium is characterized by a single parameter of the economy which measures the (inverse) moral-hazard intensity, or put simply, the debt capacity or the funding liquidity per unit of asset: (1) The moral-hazard intensity divides the set of firms into three categories – those that are fully liquidated, those that are partially liquidated, and those that provide liquidity ("arbitrageurs") and purchase assets at fire-sale prices; (2) By determining the cost of liquidating an asset relative to the cost of funding it with external finance, the moral-hazard intensity determines the equilibrium extent of de-leveraging of rationed firms; and (3) Through these first two effects, the moral-hazard intensity determines the equilibrium price at which assets are liquidated.

An interesting result that stems from this characterization is the following. As moral-hazard intensity *increases* (formally, the spread between the return on the good asset and the risk-shifting asset declines), firms' ability to raise financing against assets is lowered and equilibrium levels of liquidity in the economy fall. In turn, the market for assets clears at *lower prices*. This is simply the result that funding liquidity, measured by (inverse) moral-hazard intensity, affects market liquidity (Gromb and Vayanos, 2002 and Brunnermeier and Pedersen, 2005a). Our measure of funding liquidity is based on the amount of financing that can be raised given an agency problem tied to external finance, unlike the extant literature where it is modeled exogenously by the tightness of a margin or collateral requirement (justified as a response to some underlying agency problem).

In the preceding discussion, the ex-ante structure of liabilities undertaken by firms was treated as given. We endogenize this structure by assuming that ex ante, firms are ranked by the amount of initial financing they need to fund the project.[1] The incremental financing is raised through short-term debt contracts that give lenders the ability to liquidate ex post in case promised payments are not met. While not critical to the overall thrust of our results, we show that this short-term, rollover form of financing of assets – that grants control to lenders in case of default (as in collateral and margin requirements) – is optimal from the standpoint of raising maximum ex-ante finance.

This augmentation of our benchmark model leads to an intriguing, but somewhat involved, fixed-point problem: On the one hand, the promised payment for a given amount of financing is decreasing in the level of liquidation prices in case of default; On the other hand, the liquidation price is itself determined by the distribution of promised debt payments since these affect the ex-post rationing and de-leveraging faced by firms. We show that there is a unique solution to this fixed-point problem, characterized by the fraction of firms that are ex-ante rationed and by the mapping from moral-hazard intensity to price. In particular, depending upon the downside risk in fundamentals, which affects the ease of raising leverage, a certain fraction of poorly capitalized firms are unable to enter the financial sector. In other words, the extent of entry in the financial sector is endogenous to model parameters. The solution to the resulting fixed-point problem can be characterized by a contraction mapping and this enables us to provide a recursive, constructive algorithm for the solution.

While the endogenous nature of entry renders analytical results on comparative statics difficult, numerical examples provide valuable insights. Most strikingly, as the distribution of quality of assets improves in a first-order stochastic dominance (FOSD) sense, the distribution of moral-hazard intensity improves too, firms face weaker financing frictions and lower extent of fire sales in future, and, in turn, ex-ante lenders require lower promised payments. In other words, leverage is "cheap" and even some poorly capitalized institutions enter the financial sector. Interestingly,

[1]For example, hedge-fund managers or broker-dealers must raise different amounts of incremental financing in order to trade. This can be considered as a metaphor for differing levels of wealth or internal equity of different hedge funds.

the better ex-ante distribution of fundamentals can in fact be associated with *lower* prices when adverse shocks to asset quality materialize, compared to prices in the same ex-post states when the economy faces a worse ex-ante distribution of fundamentals.

The reason for this counterintuitive result is the endogenous nature of entry in our model. As explained above, good times in terms of expectations about the future enable even highly levered institutions to be funded ex ante. Even though bad times are less likely to follow, in case they do materialize, then the greater mass of firms that have entered the financial sector with high leverage implies that more firms end up with funding liquidity problems, are forced to de-lever through asset sales, and thus there are deeper discounts in prices.

This effect matches well the often-observed "puzzle" in financial markets that when there is a sudden, adverse asset-quality shock to the economy in a period of high expectations of fundamentals, the drop in prices seems rather severe when benchmarked in terms of drop expected from traditional, frictionless asset-pricing models. This phenomenon was highlighted in the introductory quote by Paul McCulley in PIMCO's Investment Outlook of Summer 2007 following the sub-prime crisis which seemed to have switched the financial system from one of expectations of low volatility and abundant global liquidity to one with severe asset-price correction and an equally severe drying up of liquidity. While there are many elements at work in explaining this phenomenon in the (ongoing) crisis of 2007-09, our model clarifies that financial structure of the economy as a whole, in particular, the extent of highly leveraged institutions in the system, is endogenous to expectations leading up to a crisis. This endogeneity is crucial to understanding the severity of fire sales that hit asset markets when levered institutions attempt to meet their financial liabilities.

The paper is organized as follows. Section 2 discusses the related literature. Section 3 sets up the benchmark model of risk-shifting and asset sales. Section 4 augments the benchmark model to study the ex-ante debt capacity of firms. Section 5 discusses robustness issues. Section 6 concludes. All proofs not contained in the text are provided in Appendix 1. Appendix 2 presents the constructive algorithm to solve the fixed-point problem introduced in Section 4.

2 Related literature

The idea that asset prices may contain liquidity discounts when potential buyers are financially constrained dates back to Williamson (1988) and Shleifer and Vishny (1992).[2] Since then, fire

[2] Empirically, the idea of fire sales has now found ample empirical evidence in a variety of different settings: in distressed sales of aircrafts in Pulvino (1998), in cash auctions in bankruptcies in Stromberg (2000), in creditor recoveries during industry-wide distress especially for industries with high asset-specificity in Acharya, Bharath and Srinivasan (2007), in equity markets when mutual funds engage in sales of similar stocks in Coval and Stafford (2006), and, finally, in an international setting where foreign direct investment increases during emerging market

sales have been employed in finance models regularly, perhaps most notably by Allen and Gale (1994, 1998) to examine the link between limited market participation, volatility, and fragility observed in banking and asset markets. At its roots, our model is closely linked to this literature on fire sales and industry equilibrium view of asset sales. The industry view makes clear that market prices depend on funding liquidity of potential buyers. More broadly, the overall approach and ambition of our paper in relating the distribution of liquidity needs in an economy to equilibrium outcomes is closest to the seminal paper of Holmstrom and Tirole (1998). However, there are important differences with both these sets of papers.

In Allen and Gale (1994, 1998), the liquidity shocks arise as preference shocks to depositors or investors as in Diamond and Dybvig (1983). In Holmstrom and Tirole (1998), the liquidity shocks arise as production shocks to firms' technologies. In either case, they are not endogenous outcomes. We derive liquidity needs as being determined in equilibrium by asset-liability mismatch of firms, where the level and distribution of liabilities in the economy is an outcome of model primitives such as the distribution of asset quality and moral hazard problems in future. The liabilities become liquidity "shocks" in our model in the sense that liabilities are known in advance but they take the form of "hard" debt contracts and asset quality is uncertain in future. The optimality of hard debt contract in our model with control rights given to lenders in case of default mirrors closely the work of Aghion and Bolton (1992), Hart and Moore (1994), Hart (1995), and Diamond and Rajan (2001).

In terms of modeling details, we derive limited funding liquidity as arising due to credit rationing from a risk-shifting moral hazard problem. Our specific modeling technology is closely related to the earlier models in Stiglitz and Weiss (1981) and Diamond (1989, 1991). In contrast, Holmstrom and Tirole's model of limited funding liquidity is based on rent-seeking moral hazard. It is our belief that rent-seeking is a more appropriate metaphor for agency problems affecting real or technological choices, whereas risk-substitution fits financial investment choices (typically by highly levered institutions) better.[3] Our primary goal is to consider the implications of endogenously derived funding liquidity of assets (given the risk-shifting problem) for market prices and equilibrium leverage of the financial sector.

Our work is also related to the seminal work of Kiyotaki and Moore (1997) on credit cycles. In Kiyotaki and Moore (1997) and Krishnamurthy (2003), the underlying asset cannot be pledged because of inalienable human capital.[4] However, land can be pledged and has value both as a

crises to acquire assets at steep discounts in the evidence by Krugman (1998), Aguiar and Gopinath (2005), and Acharya, Shin and Yorulmazer (2007).

[3]For instance, it is hard for an auto manufacturer to hide its risks and be doing bio-tech pursuits instead of its core business, but relatively easy for a hedge-fund manager or investment bank to hide its risks by speculating in opaque or illiquid financial assets.

[4]Krishnamurthy (2003) differs from Kiyotaki and Moore (1997) in that all contingent claims on aggregate variables are allowed subject to collateral constraints.

productive asset and as collateral. Caballero and Krishnamurthy (2001) employ a Holmstrom-Tirole approach to liquidity shocks (these are exogenous) and allow firms to post collateral in a manner similar to Kiyotaki and Moore. In contrast, the underlying asset in our model can be pledged ("asset sale") but the pledgeable amount is endogenously determined by the risk-shifting problem and the equilibrium distribution of leverage which determines the demand of assets from potential buyers. In this sense, our objectives can be considered as the financial markets counterpart to those of Bernanke and Gertler (1989) who considered the role of real collateral in ameliorating agency problems linked to real investments, and its implications for business cycle.

Our model also has implications for the recent work in finance linking market liquidity and funding liquidity due to Gromb and Vayanos (2002), Brunnermeier and Pedersen (2005a), Plantin and Shin (2006), and Anshuman and Viswanathan (2006). In Gromb and Vayanos (2002), agents can only borrow if each asset is separately and fully collateralized, i.e., borrowing is essentially riskless. In Brunnermeier and Pedersen (2005a), there is a collateral requirement that limits funding liquidity and is essentially exogenous: a shock to prices (or volatility) leads to liquidity shocks, that, in turn, leads to liquidation by financial intermediaries who engage in risk management. These models do not explicitly consider why lenders engage in risk management and why collateral requirements are imposed (even though they do recognize that agency problems must be at play). Plantin and Shin (2006) consider a dynamic variant of this feedback effect focusing on application to the unwinding of carry trades and their precipitous effect on exchange rates.[5] Anshuman and Viswanathan (2006) point out that the ability to renegotiate constraints can eliminate liquidity crises of the nature analyzed in these papers, unless some other frictions are present.

Our paper presents one such friction arising due to the ability of financial intermediaries to substitute risks, which limits their borrowing capacity. Bolton, Santos and Scheinkman (2008) consider adverse selection as the relevant friction that generates price discounts in asset liquidations and limits funding capacity of financial institutions. Both risk-shifting moral hazard and adverse selection are likely to be at play in practice. Hence, we view our work as complementary to that of Bolton et al.

Finally, in a recent paper, Lorenzoni (2007) considers a competitive equilibrium of intermediation with rent-seeking moral hazard and shows that there can be "excessive" borrowing ex ante and "excessive" volatility ex post. Lorenzoni's focus is more on the (in)efficiency of the competitive equilibrium due to the pecuniary externality of asset liquidations and on preventive policies to curb the credit bubble and improve welfare. In our model too, the pecuniary externality exists as each firm's liquidations lower asset prices, raise loss given default for lenders, and thus raise ex-ante cost of borrowing for all firms. We focus however on the positive implications for

[5] Morris and Shin (2004) present a model where traders are liquidated when an exogenous trigger price is reached and this trigger is different for each trader.

financial crises arising from a risk-shifting agency problem faced by intermediaries rather than the normative implications of the rent-seeking problem as considered by Lorenzoni.

3 Model

3.1 Informal description

Our model is set up as follows. At date 0, there is a continuum of agents who have access to identical, valuable trading technology ("asset") of limited size. Agents do not have all of the financing required to incur the fixed costs for setting up firms that will invest in this asset. Agents differ in the amount of personal initial capital they can deploy for investment. They can raise external financing from a set of financiers in order to meet the fixed costs.

Assets are specific in that financiers cannot redeploy them. In fact, we will assume assets are rendered worthless in hands of financiers unless they sell them right away to those who can deploy them. Conversely, firms are not in the business of providing external finance to each other. Some examples of this setup would be traders setting up hedge funds and borrowing from prime brokers, or broker-dealer firms (or investment banks) being set up with reliance on short-term commercial paper based financing, even though some of our assumptions make our caricature of these settings somewhat extreme.

Each asset produces an uncertain cash flow at date 2. Agents (non-financiers) have the option of switching from their asset to an alternate, riskier asset (e.g., through poor risk management of a trade) that is less valuable but may be attractive once external financing is raised. Such a switch never occurs in equilibrium but its possibility affects the nature and extent of external financing.

At date 1, an observable but non-verifiable public signal concerning the common quality of the valuable assets becomes available. If the optimal contract at date 0 so specifies, financiers may demand repayments at date 1, or they may effectively roll over their financing to date 2. An asset sale market exists where assets can be liquidated to other firms at market-clearing prices in exchange for cash that can be used to pay off existing debt. Firms acquiring assets may raise financing at date 1 against existing assets as well as assets to be acquired.

We formally specify and solve the model backwards starting with the second period between date 1 and date 2. To this end, we first assume and later prove in date-0 analysis that the optimal date-0 contract takes the form of debt that is due at date 2, but it is hard in the sense that it gives financiers (lenders) the control at date 1 to demand early repayment if it is optimal for them to do so. Taking this as an assumption to start with, we solve the second-period model for a particular realization of public information about asset quality.

3.2 Benchmark second-period model

Consider a continuum of firms that have all undertaken some borrowing at date 0. At date 1, firm i is required to pay back ρ_i to its existing creditors. Firms have no internal liquidity and must raise new external finance at date 1 to pay off existing debt. Alternatively, existing creditors can simply roll over their debt provided they are guaranteed an expected repayment of ρ_i at date 2. The contract for borrowing is hard and if the promised payment ρ_i is not met at date 1, then creditors take charge and force the firm to liquidate assets.

The time-line for the model, starting at date 1, is specified in Figure 1. All firm owners and creditors are risk-neutral and the risk-free rate of interest is zero.

After raising (new or rolled-over) external finance at date 1, there is the possibility of moral hazard at the level of each firm. In particular, we consider asset-substitution moral hazard. Firm's existing investment is in an asset which is a positive net present value investment. However, after asset sales and raising of external finance at date 1, each firm can switch its investment to another asset.

We denote the assets as j, $j \in \{1,2\}$, yielding a date-2 cash flow per unit size of $y_j > 0$ with probability $\theta_j \in (0,1)$, and no cash flow otherwise. We assume that $\theta_1 < \theta_2$, $y_1 > y_2$, $\theta_1 y_1 \leq \theta_2 y_2$, and $\theta_1 y_1 \leq \rho_i$. In words, the first asset is riskier and has a higher payoff than the second asset, but the second asset has a greater expected value. Also, taking account of the financial liability at date 1, investing in the first asset is a negative net present value investment for all firms. We assume the shift between assets is at zero cost. The simplest interpretation could be a deterioration in the risk-management function of the financial intermediary or outright fraud, that allows pursuit of riskier strategies with the same underlying asset or technology. We discuss some other possibilities in Section 5.

The external finance at date 1 is raised in the form of debt with face value of f to be repaid at date 2. Then, the incentive compatibility condition to ensure that firm owners invest in asset $j=2$ (that is, do not risk-shift to asset $j=1$) requires that

$$\theta_2(y_2 - f) > \theta_1(y_1 - f). \tag{1}$$

This condition simplifies to an upper bound on the face value of new debt:

$$f < f^* \equiv \frac{(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} \ . \tag{2}$$

Since this condition bounds the face value of debt that can provide incentives to invest in the better asset, we obtain credit rationing as formalized in the following lemma. We acknowledge that this result is by itself not new (see, for example, Stiglitz and Weiss, 1981).

Lemma 1 *Firms with liability of ρ at date 1 that is greater than $\rho^* \equiv \theta_2 f^*$ cannot roll over debt by only issuing new external finance; that is, they are credit-rationed.*

To see this result, note first that $f^* < y_2$ so that borrowing up to face value f^* is indeed feasible in equilibrium provided it enables the borrowing firm to meet its funding needs. In other words, firms with $\rho \leq \rho^* \equiv \theta_2 f^*$ borrow, invest in the better asset, and simultaneously meet their funding constraint. Second, note that for $\rho > \rho^*$, investment is in the first, riskier asset. However, in this case funding constraint requires that the face value be $\hat{f} = \frac{\rho}{\theta_1}$ which is greater than y_1 for all $\rho > \rho^*$. That is, firms with liability ρ exceeding ρ^* cannot borrow and are rationed.

We assume in what follows that the continuum of firms is ranked by liabilities ρ such that $\rho \sim g(\rho)$ over $[\rho_{\min}, \rho_{\max}]$, where $\rho_{\min} \equiv \theta_1 y_1 < \theta_2 y_2 \leq \rho_{\max}$ and $\rho^* \in [\rho_{\min}, \rho_{\max}]$. Thus, Lemma 1 implies that firms in the range $(\rho^*, \rho_{\max}]$ are credit-rationed in our benchmark model and must "de-lever", that is, engage in asset sales to pay off some or all of their existing debt.

3.3 Asset sales

Suppose a firm can sell its assets at market-clearing price of p, which we endogenize later. If firm sells α units of assets, it generates αp as proceeds from asset sale which can be used to repay its debt. The remaining balance-sheet of the firm is of the size $(1 - \alpha)$, and its per unit debt capacity is ρ^* as in Lemma 1. Thus, if it sells α units of assets, its funding liquidity is given by $[\alpha p + (1 - \alpha)\rho^*]$. As long as liquidation price p exceeds the per unit debt capacity of the risky asset ρ^*, funding liquidity expands with asset sales. We assume and show later that it is indeed the case that $p \geq \rho^*$.

Since the firm needs to raise ρ units in total to roll over its debt, it must choose a liquidation policy $\alpha \geq 0$ such that

$$\rho \leq [\alpha p + (1 - \alpha)\rho^*] \,. \tag{3}$$

For firms with $\rho < \rho^*$, this constraint is met without engaging any asset sales. For rationed firms of Lemma 1, that is, for $\rho > \rho^*$, we obtain the following result:

Proposition 1 *If the liquidation price p is greater than ρ^*, then asset sales relax credit rationing for firms with $\rho \in (\rho^*, p]$, and firm with liability ρ engages in asset sale of α units, where*

$$\alpha(\rho) = \frac{(\rho - \rho^*)}{(p - \rho^*)} \,. \tag{4}$$

Thus, asset sales by a firm are increasing in its liability ρ and decreasing in liquidation price p. Finally, the proportion of firms for which credit rationing is relaxed, $[p - \rho^*]$, is increasing in liquidation price p.

The liquidation price p plays a crucial role in determining the extent of asset sales or deleveraging. In particular, if liquidation price is low, then firms have to liquidate a large part of their existing investment. Also, if liquidation price is higher then more firms that were otherwise rationed can be funded in equilibrium with asset sales.

Next, we introduce a market for liquidation of the asset at date 1 and study how it influences and is influenced by the equilibrium level of asset sales. Also, we assumed in the analysis above that $p \leq \rho_{\max}$. We verify below that this will indeed be the case under our maintained assumption $\theta_2 y_2 \leq \rho_{\max}$.

3.4 Market for asset sales

Assets liquidated by firms that face rationing ($\rho > \rho^*$) are acquired by those that are not rationed ($\rho < \rho^*$) and have spare debt capacity. We consider standard market clearing for asset sales. An important consideration is that asset purchasers, by virtue of their smaller liabilities, may be able to raise liquidity not only against their existing assets but also against to-be-purchased assets.

Formally, suppose that a non-rationed firm with liability ρ acquires α additional units of assets. Then, the total amount of liquidity available for asset purchase with such a non-rationed firm is given by

$$l(\alpha, \rho) = [(1+\alpha)\rho^* - \rho] \, . \tag{5}$$

That is, the funding ability of a non-rationed firm consists of its spare debt capacity from existing assets, $(\rho^* - \rho)$, plus the liquidity that can be raised against assets to be acquired, $\alpha\rho^*$.

The pertinent question is: How many units of assets would this firm be prepared to buy as a function of the price p? Note that no firm would acquire assets at a price higher than their expected payoff (under the better asset). Denoting this price as $\overline{p} = \theta_2 y_2$, we obtain the following demand function $\hat{\alpha}(p, \rho)$ for the firm. For $p > \overline{p}$, $\hat{\alpha} = 0$. For $p < \overline{p}$, $\hat{\alpha}$ is set to its highest feasible value given the liquidity constraint:

$$p \; \hat{\alpha} = l(\hat{\alpha}, \rho), \tag{6}$$

which simplifies to

$$\hat{\alpha}(p, \rho) = \frac{(\rho^* - \rho)}{(p - \rho^*)}. \tag{7}$$

Finally, for $p = \overline{p}$, buyers' demand is indifferent between 0 and $\hat{\alpha}$ (evaluated at $\overline{p}$).

Thus, the total demand for assets for $p < \overline{p}$ is given by

$$D(p, \rho^*) = \int_{\rho_{\min}}^{\rho^*} \hat{\alpha}(p, \rho) g(\rho) d\rho = \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)} \, g(\rho) d\rho, \tag{8}$$

where we have stressed the dependence on (inverse) moral hazard intensity ρ^*.

Given this demand function for non-rationed firms, we can specify the market-clearing condition. Note that the total supply of assets up for liquidation is given by

$$S(p, \rho^*) = \int_{\rho^*}^{p} \frac{(\rho - \rho^*)}{(p - \rho^*)} \, g(\rho) d\rho + \int_{p}^{\rho_{\max}} g(\rho) d\rho. \tag{9}$$

The two terms correspond respectively to (i) partial asset liquidations by firms with $\rho \in (\rho^*, p]$ to meet their liabilities, and (ii) complete liquidation of firms with $\rho \in (p, \rho_{\max}]$ which cannot fully meet their liabilities.

Then, the equilibrium price p^* satisfies the market-clearing condition

$$D(p, \rho^*) = S(p, \rho^*). \tag{10}$$

In particular, if excess demand is positive for all $p < \overline{p}$, then $p^* = \overline{p}$ (since the buyers are indifferent at this price between buying and not buying, and hence their demand can be set to be equal to the supply).

Before characterizing the behavior of the equilibrium price, it is useful to consider properties of the demand and supply functions. First, both demand and supply functions decline in price p. This is because as price increases, asset purchasers can only buy fewer assets given their limited liquidity. Simultaneously, rationed firms need to liquidate a smaller quantity of their assets. Hence, what is important is the behavior of excess demand function, $E(p, \rho^*) \equiv [D(p, \rho^*) - S(p, \rho^*)]$, as a function of price p. We focus below on the case where $p < \overline{p}$, relegating the details of the case where $p = \overline{p}$ to the Appendix (in Proof of Proposition 2).

The excess demand function can be rewritten as:

$$\begin{aligned} E(p, \rho^*) &= D(p, \rho^*) - S(p, \rho^*) && (11) \\ &= \int_{\rho_{\min}}^{p} \frac{(\rho^* - \rho)}{(p - \rho^*)} \, g(\rho) d\rho - \int_{p}^{\rho_{\max}} g(\rho) d\rho. && (12) \end{aligned}$$

Integrating this equation by parts yields

$$E(p, \rho^*) = -1 + \frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^{p} G(\rho) d\rho \tag{13}$$

where $G(\rho) = \int_{\rho_{\min}}^{p} g(\rho) d\rho$ and $G(\rho_{\min}) = 0$.

The condition that excess demand be zero, i.e., $E(p, \rho^*) = 0$, leads to the relationship

$$p = \rho^* + \int_{\rho_{\min}}^{p} G(\rho) d\rho. \tag{14}$$

If the solution to this equation exceeds $\overline{p}$, then we have $p^* = \overline{p}$.

From this representation of market-clearing condition, we observe that the price can never fall below the threshold level of ρ^* (as we assumed earlier while deriving Proposition 1). This is because non-rationed firms can always raise ρ^* of liquidity against each additional unit of asset they purchase. Hence, at $p = \rho^*$, their demand for asset purchase is infinitely high. The second term captures the effect of spare liquidity in the system. Intuitively, if this spare liquidity is high, then the price is at its frictionless value of $\overline{p}$, else it reflects a fire-sale discount.

Second, the price can never be higher than $\overline{p}$ as above this price, demand is zero and there can be no market clearing. Together, these two facts guarantee an interior market-clearing price $p^* \in [\rho^*, \overline{p}]$.

Third, as intuition would suggest, the excess demand function is decreasing in price p, which gives us that p^* is in fact unique.

And, finally, the key determinant of the market-clearing price is the extent of (inverse) moral hazard intensity ρ^*. This is the central parameter that drives all action in the model: It determines the partition of firms into rationed firms and non-rationed firms, the extent of buying power of non-rationed firms, and, also, the extent of asset liquidations.

The resulting equilibrium price satisfies the following proposition:

Proposition 2 *The market-clearing price for asset sales, p^*, is unique and weakly increasing in the (inverse) moral hazard intensity ρ^*, which is also the debt capacity per unit of the asset, in the following manner:*

(i) There exists a critical threshold $\hat{\rho}^* < \overline{p}$ such that $p^* = \overline{p}, \forall \rho^* \geq \hat{\rho}^*$; and,

(ii) For $\rho^* < \hat{\rho}^*$, $p^* \in [\rho^*, \overline{p})$, p^* is strictly increasing in ρ^*, and $p^* = \rho^*$ only when $\rho^* = \rho_{\min}$. Therefore, in this region, there is an illiquidity discount, $[\overline{p} - p^*]$, whose size is declining in ρ^*.

When ρ^* is above a critical value $\hat{\rho}^* > \rho_{\min}$, assets are liquidated at their highest valuation: Few firms are rationed, buyers (non-rationed firms) have lot of liquidity and sellers (rationed firms) do not need to de-lever much. As the moral hazard problem becomes worse, that is, ρ^* declines, there is not enough liquidity in the system to absorb the pool of assets being put up for liquidation at the highest price. Hence, the market-clearing price is lower than $\overline{p}$. Since assets are "cheap", non-rationed firms demand as much as possible of the liquidated assets with their entire available liquidity. On the supply side, as price falls, more firms are rationed, and rationed firms must liquidate more. As the moral hazard problem keeps worsening (ρ^* becomes smaller), prices fall until they hit ρ^* eventually, and this happens when in fact ρ^* equals $\rho_{\min}$.

Note that the liquidation price exhibits "cash-in-the-market pricing" as in Allen and Gale (1994, 1998) since it depends on the overall amount of liquidity available in the system for asset purchase, which, in turn, is determined by the extent of moral hazard problem. The important message from this analysis is that whether a rationed firm can relax its own borrowing constraint or not by selling assets depends upon the liquidity of the potential purchasers of its assets (through the liquidation price) and on the liquidation of assets by other such rationed firms. The moral hazard parameter ρ^* partitions firms endogenously into liquidity providers and takers, based on the magnitude of their liquidity shocks, and one can think of the excess demand for the asset, $E(p,\rho^*) \equiv [D(p,\rho^*) - S(p,\rho^*)]$, given by equation (12), as an inverse measure of the *excess financial leverage in the system*.[6]

Another important observation is that part (ii) of Proposition 2 implies a natural link between funding liquidity of firms and liquidity of asset markets. Funding liquidity in our model is measured by ρ^*, the amount of financing that can be raised per unit of asset. Market illiquidity in our model can be measured as the fire-sale discount in prices, $[\overline{p} - p^*]$. The Proposition formally shows that funding liquidity and market illiquidity are negatively related. While the link here is only from funding liquidity to market liquidity, our augmented model of Section 4 will also formalize the reverse link from market liquidity to (ex-ante) funding liquidity. Unlike the extant literature where funding liquidity is modeled through exogenously specified margin or collateral requirements, our measure of funding liquidity is linked to the amount of financing that can be raised given the risk-shifting problem tied to leverage. Formally, it is given by ρ^*. This linkage is quite important in the analysis to follow.

Reverting to our current model, we combine Proposition 2 with Proposition 1 to obtain the following natural result that the extent of asset sales required by a rationed firm is higher when the moral hazard problem is more severe.

Proposition 3 *The extent of asset sale by firm with liability ρ, denoted as $\alpha(\rho)$, is decreasing in the (inverse) moral hazard intensity ρ^* which is also the debt capacity per unit of the asset.*

The following example which assumes a uniform distribution on the liabilities helps us illustrate these equilibrium relationships graphically.

[6]These features of our model are essentially variants of the industry-equilibrium effects in Shleifer and Vishny (1992)'s model. Crucially, however, the determinant of rationing and of the limited ability of buyers to purchase are both tied to the same underlying state variable, the extent of moral hazard problem.

Example: Suppose that $\rho \sim Unif[\rho_{\min}, \rho_{\max}]$ and $\overline{p} = \theta_2 y_2 = \rho_{\max}$. Then, solving the market-clearing condition $E(p, \rho^*) = 0$, yields the following equilibrium relationships:

1. If $\rho^* \geq \hat{\rho}^* \equiv \frac{1}{2}(\rho_{\min} + \rho_{\max})$, then the price for asset sales is $p^* = \rho_{\max}$;

2. Otherwise, that is, if $\rho^* < \frac{1}{2}(\rho_{\min} + \rho_{\max})$, then there is cash-in-the-market pricing and the price for asset sales is

$$p^* = \rho_{\max} - \sqrt{(\rho_{\max} - \rho_{\min})}\sqrt{(\rho_{\max} + \rho_{\min} - 2\rho^*)}.$$

3. In the cash-in-the-market pricing region, the equilibrium price p^* is increasing and convex in (inverse) moral hazard intensity ρ^*. In particular,

$$\frac{dp^*}{d\rho^*} = \frac{\sqrt{(\rho_{\max} - \rho_{\min})}}{\sqrt{(\rho_{\max} + \rho_{\min} - 2\rho^*)}} \ > \ 0,$$

and

$$\frac{d^2 p^*}{d\rho^{*2}} = \sqrt{(\rho_{\max} - \rho_{\min})}(\rho_{\max} + \rho_{\min} - 2\rho^*)^{-\frac{3}{2}} \ > \ 0.$$

4. The asset sale function $\alpha(\rho)$ is given accordingly by Proposition 1 and the expressions for liquidation price p^* in the two regions (Points 1 and 2 above).

The price p^* and the amount of leverage repaid, that is, asset sale proceeds $\alpha(\rho)p$, are illustrated in Figures 2 and 3. Figure 2 shows the cash-in-the-market pricing in asset market when funding liquidity is below $\hat{\rho}^*$. Figure 3 in particular is striking.[7] As the moral hazard problem worsens (ρ^* falls), a smaller range of firms is able to relax rationing and at the same time these firms face increasingly greater de-leveraging. Finally, Figure 4 plots market illiquidity, measured as the fire-sale discount in asset price, $[\overline{p} - p^*]$, as a function of the funding liquidity per unit of asset, ρ^*. It illustrates that when funding liquidity is high, market liquidity is at its maximal level. As funding liquidity deteriorates and falls below $\hat{\rho}^*$, market becomes illiquid and increasingly so as funding liquidity deteriorates.

Interpretation of moral hazard intensity: What does it mean to vary the moral hazard parameter ρ^*? Recall that $\rho^* = \frac{\theta_2(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}$, so that ρ^* is increasing in θ_2, the quality of the better asset. Thus, a decrease in ρ^* can be given the economically interesting interpretation of a deterioration in the quality of assets, for example, over the business cycle. Note that we are

[7] The parameters in Figure 3 are: $\theta_2 = 0.8$, $y_2 = 12.5$, giving $\rho_{\max} = 10$, and $\theta_1 = 0.2$, $y_1 = 20$, giving $\rho_{\min} = \theta_1 y_1 = 4$.

holding constant the quality of bad asset θ_1. So strictly speaking, if the better asset deteriorates in quality in a relative sense compared to the other asset during a business-cycle downturn, then the moral hazard problem gets aggravated. Thus, our model entertains a natural interpretation that during economic downturns and following negative shocks to the quality of assets, there is greater credit rationing and de-leveraging in the economy. Accompanying these are lower prices for asset liquidations due to the deterioration in asset quality and the coincident deterioration in funding liquidity.

In our analysis so far, we assumed the distribution of liabilities was unrelated to the quality of assets. Relaxing this would formally imply a relationship between θ_2 and the distribution of liquidity shocks $g(\rho)$. We explore and build this link in Section 4 where we introduce and analyze the ex-ante date-0 structure of the model.

4 Ex-ante debt capacity

In this section, we provide an equilibrium setting that gives rise to the structure of liabilities ρ_i assumed in our model so far. Before we move to modeling details, we provide a summary of what this section achieves.

We endogenize the structure of liabilities in Section 4.1 by assuming that ex ante (at date 0), firms are ranked by their initial wealth or capital levels and must raise incremental financing up to some fixed, identical level in order to trade. The incremental financing is raised through short-term debt contracts, payable at date 1. The contracts give lenders the ability to liquidate ex post in case promised payments are not met. We show in Section 4.4 that this form of financing – which grants control to lenders in case of default (as in collateral and margin requirements) – is optimal from the standpoint of raising maximum ex-ante finance.

This augmentation of the benchmark model leads to an interesting, even if somewhat involved, fixed-point problem: On the one hand, the promised payment for a given amount of financing is decreasing in the level of liquidation prices in case of default; on the other hand, the liquidation price is itself determined by the distribution of promised debt payments to be met by firms. We show in Section 4.2 that there is a unique solution to this fixed-point problem, characterized by the fraction of firms that are ex-ante rationed (that is, firms that are unable to raise enough debt to meet the fixed costs) and the ex-post mapping from moral-hazard intensity to price. In fact, the fixed-point is a contraction mapping and enables us to provide a recursive, constructive algorithm for the solution (provided in Appendix 2). While the ex-ante rationing of firms renders analytical results on comparative statics difficult, numerical examples in Section 4.3 confirm some conjectures that follow naturally from our analysis.

4.1 The set-up

The augmented time-line is specified in Figure 5.

Suppose that at date 0, there is a continuum of firms that have access to an investment opportunity that has identical payoffs. However, each firm has to finance a different amount. We assume that this investment shortfall s_i is externally financed via a debt contract with a fixed, promised payment of ρ_i at date 1, against which creditors provide financing of s_i; the ex-ante cumulative distribution function of s_i is given by $R(s_i)$ over $[s_{\min} = \theta_1 y_1, s_{\max}]$. This assumption on the range of s_i ensures that no debt less than the value of the bad project is issued.

The investment opportunity can yield in two periods (date 2) a cash flow y_2 with probability θ_2. However, after issuance of rollover debt and asset sales at date 1, there is the possibility of moral hazard: Firm owners, if optimal to do so, may switch from the existing safer asset to the riskier asset, which yields a cash flow y_1 with probability θ_1, where we we assume as in our benchmark model that $\theta_1 < \theta_2$, $y_1 > y_2$, and $\theta_1 y_1 < \rho_i < \theta_2 y_2$. Viewed from date 0, θ_2 is uncertain: θ_2 has cumulative distribution function (cdf) $H(\theta_2)$ and probability density function (pdf) $h(\theta_2)$ over $[\theta_{\min}, \theta_{\max}]$, where we assume for simplicity that $\theta_{\min} y_2 \geq \theta_1 y_1$, that is, the worst-case expected outcome for the safer asset is no worse than that for the riskier asset. In fact we impose that

$$\theta_{\min} = \frac{\theta_1 y_1}{y_2} \left[1 + \sqrt{1 - \frac{y_2}{y_1}} \right]. \tag{15}$$

This assumption ensures that maximum amount that can be borrowed is determined by ρ^* (which is always higher than $\theta_1 y_1$).

Firms can attempt to meet the promised payment ρ_i by rolling over existing debt or issuing new debt. Firms may also de-lever by selling assets. Note that ρ_i is fixed in that it is not contingent on the realization of θ_2, which we assume is observable but not verifiable. If the payment ρ_i cannot be met at date 1 , then there is a transfer of control to creditors who liquidate the assets and collect the proceeds.

Thus, the date-1 structure of this augmented model maps one for one (for a given realization of θ_2) into the date-1 structure in our benchmark model where liabilities were taken as given. In particular, the lower the realization of θ_2, the lower is the per unit debt capacity of the asset at date 1, denoted as $\rho^*(\theta_2)$, and hence, the greater is the moral hazard problem; thus θ_2 indexes fundamental information that is related to the severity of the moral hazard problem.

We show next that the distribution of investment shortfall s_i at date 0 translates into an equilibrium distribution of corresponding promised debt payments ρ_i. Consider a particular realization of the quality of investment opportunity, say θ_2, at date 1. As shown in Proposition 1, firms with liabilities up to $\rho^*(\theta_2) = \theta_2 f^*(\theta_2) = \frac{\theta_2(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}$ are not rationed. These firms can meet their outstanding debt payments at date 1, continue their investments, and possibly, also acquire more assets. Next, as shown in Proposition 1, firms with liabilities in the range $[\rho^*(\theta_2), p^*(\theta_2)]$ are able to meet their debt payments but only by de-leveraging through asset sales. In other words, these firms can also meet their outstanding debt payments at date 1 and continue their investments, but do not have spare liquidity to acquire more assets. Finally, firms with liabilities greater than $p^*(\theta_2)$ cannot meet their outstanding debt payments, and creditors liquidate these firms' assets.

Then, since date-0 creditors are risk-neutral, the amount of financing s_i that firm i can raise at date 0, satisfies their individual rationality constraint:

$$s_i = \int_{\theta_{\min}}^{p^{*-1}(\rho_i)} p^*(\theta_2)h(\theta_2)d\theta_2 + \int_{p^{*-1}(\rho_i)}^{\theta_{\max}} \rho_i h(\theta_2)d\theta_2 \ , \tag{16}$$

which captures the fact that for low realizations of θ_2, the moral hazard is severe and at least some firms end up being rationed, unable to meet their debt payments, and thus, liquidated, whereas for high realizations of θ_2, debt payments are met. The critical threshold determining whether θ_2 realization is "low" or "high" for firm i is given implicitly by the relation: $\rho_i = p^*(\theta_2)$. Also implicit in Equation (16) is the fact that some low wealth borrowers may be excluded as the amount owed s_i may not be covered by the maximum amount available for payment the next period.

Note that given a price function $p^*(\theta_2)$ and financing s_i, equation (16) gives the face value ρ_i directly. However, we need to take account of Proposition 2 and recognize that the market-clearing price $p^*(\theta_2)$ itself depends upon the entire distribution of liabilities ρ_i across firms. In case a firm is in default, creditors recover an amount that depends upon the asset liquidation price, and, thus on the liabilities of *other* firms; in turn, each firm's ex-ante debt capacity depends on the expectation over the amount recovered. Thus the model can be viewed as a general equilibrium version of Shleifer and Vishny (1992) with ex-post as well as ex-ante contracting and endogenous borrowing capacity of firms determined by the risk-shifting moral hazard problem.

With this background, we define the equilibrium of the ex-ante game. An important notational issue to bear in mind is that in the benchmark model, we assumed as exogenously given the distribution of liabilities, $G(\rho)$, but in the augmented model, this distribution is induced by the distribution of financing needs, $R(s)$.

Definition: An equilibrium of the ex-ante borrowing game is (i) a pair of functions $\rho(s_i)$ and $p^*(\theta_2)$, which respectively give the promised face-value for raising financing at date 0, s_i, and the equilibrium price given quality of assets θ_2; and (ii) a truncation point $\hat{s}$, which is the maximum amount of financing that a firm can raise at date 0, such that $\rho(s_i)$, $p^*(\theta_2)$ and $\hat{s}$ satisfy the following fixed-point problem.

1. For every θ_2, price is determined by the industry equilibrium condition of Proposition 3:

$$p^*(\theta_2) \leq \rho^*(\theta_2) + \int_{\rho_{\min}}^{p^*(\theta_2)} \hat{G}(u)du \ , \tag{17}$$

where compared to equation (14), we have replaced distribution of liabilities $G(\cdot)$ with the distribution $\hat{G}(\cdot)$ and also substituted the variable of integration ρ with u to avoid confusion with the function $\rho(s_i)$. In particular, $\hat{G}(u)$ is the truncated equilibrium distribution of liabilities given by $\hat{G}(u) = \frac{R(\rho^{-1}(u))}{R(\hat{s})}$. Formally, $\hat{G}(u)$ is induced by the distribution of financing amounts, $R(s)$, via the function Prob$[\rho(s_i) \leq u | s_i \leq \hat{s}]$. As in case of equation (14), a strict ($<$) inequality in equation (17) leads to $p^*(\theta_2) = \overline{p}(\theta_2) = \theta_2 y_2$.

2. Given the price function $p^*(\theta_2)$, for every $s_i \in [0, \hat{s}]$, the face value ρ is determined by the requirement that lenders receive in expectation the amount that is lent:

$$s_i = \int_{\theta_{\min}}^{p^{*-1}(\rho)} p^*(\theta_2)h(\theta_2)d\theta_2 + \int_{p^{*-1}(\rho)}^{\theta_{\max}} \rho h(\theta_2)d\theta_2. \tag{18}$$

3. The truncation point $\hat{s}$ for maximal financing is determined by the condition

$$\hat{s} \leq \int_{\theta_{\min}}^{\theta_{\max}} p^*(\theta_2)h(\theta_2)d\theta_2 \ , \tag{19}$$

with a strict inequality implying that $\hat{s} = s_{\max}$ (all borrowers are financed).

For future reference, we note that differentiating equality versions of Equations (17) and (18) yields alternative but equivalent conditions that

$$\frac{dp}{d\theta_2} = \frac{\frac{d\rho^*(\theta_2)}{d\theta_2}}{1 - \hat{G}(p)} \ \text{ if } p < \theta_2 y_2, \ \text{ else } \frac{dp}{d\theta_2} = y_2 \ , \tag{20}$$

and

$$\frac{d\rho}{ds_i} = \frac{1}{1 - H(p^{*-1}(\rho))} \ \text{ if } \rho \geq p^*(\theta_{\min}), \ \text{ else } \frac{d\rho}{ds_i} = 1 \ . \tag{21}$$

4.2 The solution

We show that there is a unique equilibrium to the ex-ante borrowing game given by the solution to the fixed-point problem stated above. We provide an explicit characterization of the solution. In what follows, we suppress the subscript i unless it is necessary.

It is easier to analyze the fixed-point problem by working with the inverse functions $s(\rho)$ and $\theta_2(p)$. $s(\rho)$ gives the financing raised ex ante for a given face-value ρ while $\theta_2(p)$ gives the realization of the state θ_2 for the given equilibrium price p. Since these are one-to-one functions, we can follow this approach. Notice that both ρ and p have the domain $[\theta_1 y_1, \theta_{\max} y_2]$ (one cannot have a face value higher than the highest possible price); it is possible that the upper bound is not reached in equilibrium and we will account for this.

The fixed point problem can be solved as follows. Fix a maximal financing $\hat{s}$. First we invert Equation (17) and solve for $\theta_2(p)$: We show below that since this is an explicit quadratic equation, we can solve for this variable. We impose the constraint that price is at most $\theta_2 y_2$. We can substitute $\theta_2(p)$ into the differential equation for $s(\rho)$, equation (21), to obtain an *integro-differential* equation that has a unique solution for $s(\rho)$. The maximum financing is then uniquely solved by the boundary condition in Equation (19).

Given the cdf of amount financed, $R(s)$, the cdf of face values conditional on financing being over the truncated support of amounts financed $[\theta_1 y_1, \hat{s}]$, is denoted as $\hat{G}(u)$, and is given by $\hat{G}(u) = \frac{R(s(u))}{R(\hat{s})}$, where $\hat{G}(u) = \text{Prob}[\rho \leq u | s \leq \hat{s}] = \text{Prob}[s(\rho) \leq s(u) | s \leq \hat{s}]$.

Define

$$L(p) = p - \int_{\theta_1 y_1}^{p} \hat{G}(\rho) d\rho, \tag{22}$$

where we have switched back to ρ as being the variable of integration.

Then, setting $L(p) = \rho^*(\theta_2)$ to satisfy equation (17) with equality, we obtain

$$\theta_2 \frac{(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} = L(p) \ , \tag{23}$$

which yields the following solution for θ_2 (we have a quadratic equation and pick the correct root)

$$\frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4 y_2 L(p) \theta_1}}{2 y_2} \ . \tag{24}$$

Accounting for the fact that prices cannot be above $\theta_2 y_2$ (hence $\theta_2 \geq \frac{p}{y_2}$), we define $\theta_2(p)$ implicitly in terms of $s(\rho)$ as:

$$\theta_2(p) = \max \left\{ \frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4 y_2 L(p) \theta_1}}{2 y_2}, \frac{p}{y_2} \right\} \tag{25}$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$. Note that this equation defines $\theta_2(p)$ in terms of $s(\rho)$ since $L(p)$ depends on the function $\hat{G}(\rho) = \frac{R(s(\rho))}{R(\hat{s})}$.[8]

[8]Note that if $p = \theta_1 y_1$, then Equation (25) is determined by Equation (24) and $\theta_2(p) = \theta_{\min}$ as $L(\theta_1 y_1) =$

Next, we solve the differential equation implied by Equation (21) (which is itself equivalent to Equation 18):

$$\frac{ds}{d\rho} = 1 - H(\theta_2(\rho)), \tag{26}$$

where $H(\theta_2)$ is the cdf of θ_2. Since it is possible that $\theta_2(p) > \theta_{max}$ in Equation (26), we extend $H(\theta_2)$ by assuming that $H(\theta_2) = 1$ for $\theta_2 > \theta_{\max}$ (this is true and innocuous since $1 - H(\theta_2) = 0$ for such θ_2).

Substituting for $\theta_2(p)$ from equation (25), we obtain that

$$\frac{ds}{d\rho} = 1 - H\left(\max\left\{\frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2}\right\}\right) \tag{27}$$

with the end-point constraint that $s(\theta_1 y_1) = \theta_1 y_1$.

This is a standard integro-differential equation of the form

$$\frac{ds}{d\rho} = f\left(\rho, \int_{\theta_1 y_1}^{\rho} \frac{R(s(u))}{R(\hat{s})} du\right) \tag{28}$$

with the end-point constraint $s(\theta_1 y_1) = \theta_1 y_1$, and it has a unique solution if the function $f(\rho, t)$ is Lipschitz in t and the function $R(s)$ is Lipschitz in s.[9] This is indeed the case in our set-up, technical details of which are relegated to Appendix 1.

We now solve for the maximal financing $\hat{s}$, which is given by the condition

$$\hat{s} \leq \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2) h(\theta_2) d\theta_2 \tag{29}$$

where $p(\theta_2)$ is the inverse function of $\theta_2(p)$ and $h(\theta_2)$ is the density of θ_2.

The left hand side of Equation (29) is $\theta_1 y_1$ at $\hat{s} = \theta_1 y_1$ and increasing in $\hat{s}$. The right hand side of Equation (29) is strictly greater than $\theta_1 y_1$ at $\hat{s} = \theta_1 y_1$ and decreasing in $\hat{s}$.[10] Either Equation (29) has a unique solution or no solution with strict inequality at $\hat{s}$, in that case there is no exclusion and $\hat{s} = I - \theta_1 y_1$.

$\theta_1 y_1$. At the other end point, $p = \theta_{\max} y_2$, either we have $\theta_2(\theta_{\max} y_2) = \theta_{\max}$, and there is no price discount at $\theta_{\max}$; or $\theta_2(\theta_{\max} y_2) > \theta_{\max}$ and there is a price discount in every state.

[9] More details of this proof (we follow Theorem 2.1 from Granas and Dugundji (2003)) are in Appendix 1. Note that the generic function f for expressing the integro-differential equation is not to be confused with the face-value of debt in our benchmark model.

[10] To see this note that if we increase $\hat{s}$, we decrease $\hat{G}(\rho)$, which means we increase $L(p)$ and hence $\theta_2(p)$; therefore $p(\theta_2)$ decreases, and, in turn, the right hand side of the Equation (29) decreases.

This completes the proof that a solution to the fixed-point problem exists and is unique.

We state all this in the following proposition.

Proposition 4 *There exists a unique equilibrium to the ex-ante borrowing game defined in Section 4.1. In particular, given a maximal borrowing amount $\hat{s}$, the borrowing function $s(\rho)$ (financing as a function of face value borrowed) is the unique solution to the integro-differential equation*

$$\frac{ds}{d\rho} = 1 - H\left(\max\left\{\frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2}\right\}\right) \tag{30}$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$. Given $s(\rho)$, the inverse equilibrium price function $\theta_2(p)$ is uniquely given by

$$\theta_2(p) = \max\left\{\frac{(\theta_1 y_1 + L(p) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4y_2 L(p)\theta_1}}{2y_2}, \frac{p}{y_2}\right\} \tag{31}$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$.

The maximal borrowing amount is uniquely given by the boundary condition

$$\hat{s} \leq \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2) h(\theta_2) d\theta_2 \tag{32}$$

where $p(\theta_2)$ is implicitly a function of $\hat{s}$.

In fact, the solution to the fixed-point problem between promised debt payments and liquidation price is a contraction and can be computed using a recursive algorithm that we outline in Appendix 2.

4.3 Numerical examples

The comparative statics with respect to a change in the distribution of financing amount s and a change in the distribution of fundamentals θ are in general ambiguous in our model because of the effect of endogenous entry (the last marginal project that can be financed varies with parameters). If we keep the set of firms that are financed at date 0 fixed, then the comparative statics are easily obtained. However, an improvement in the expectation of fundamentals (for example, a first-order stochastic dominance (FOSD) increase in distribution of θ) has two effects.

The first effect is to weakly increase prices at date 1, for a given pool of firms financed at date 0. This increase in prices lowers the cost of debt that results in the pool of firms financed at date 0 to expand so as to also include higher leverage firms. We show below that this latter effect means that at low realizations of fundamentals (which are less likely given the FOSD increase), prices can sometimes be lower with better ex-ante expectation of fundamentals.

Note that we do not have an explicit role for "volatility" in the model. Since a better distribution of asset quality leads to lower defaults in the model, our comparative static could be interpreted to some extent as delivering results one would get with low versus high volatility of news about the asset quality. But perhaps a more accurate description of our comparative static exercise is that it is about "credit risk" or "downside risk".

To understand these effects further, we solve two numerical examples using the recursive algorithm provided in Appendix 2 to compute the equilibrium. In both numerical examples, we consider a situation where the distribution of quality of asset improves in a FOSD sense, and, in turn, so does the (inverse) moral-hazard intensity.

Varying the distribution of moral-hazard intensity

Our first numerical example provides some counterintuitive insights and is constructed as follows:

1. Let $s_{\min} = \theta_1 y_1 = 0.2$, $s_{\max} = 1$, $y_1 = 4$, $y_2 = 1$, $\theta_1 = 0.05$.
   Hence s has support $[0.2, 1]$.

2. Let $t = 1 - 0.2 = 0.8$ (which is also the value of $s_{\max} - s_{\min}$) and suppose that

$$R(s) = \frac{s - 0.2}{t}, \tag{33}$$

   which is the uniform distribution. We suppose that $H(\theta)$ is given by the following distribution on $[\theta_{\min}, \theta_{\max}]$:

$$H(\theta) = 1 - \left(1 - \frac{\theta - \theta_{\min}}{\theta_{\max} - \theta_{\min}}\right)^{1/\gamma}, \tag{34}$$

   where γ, $\gamma > 0$ (note that $\gamma = 1$ corresponds to the uniform distribution). A higher value of γ implies first-order stochastic dominance (FOSD); in fact for any truncation $\hat{s}$, a higher value of γ implies FOSD.[11] Also, note that $E[\theta]$ is $\theta_{\min} + \frac{(\theta_{\max} - \theta \min)\gamma}{1+\gamma}$ which is increasing in γ.

[11] Hopenhayn (1992) refers to this as monotone conditional dominance or MCD.

We let γ take values in $\{0.5, 5.0\}$. We show for these values the distributions of $\rho(s)$ in one plot (Figure 6) and $p(\theta)$ in another plot (Figure 7). The figures show large variations in prices and hence large variations in liabilities (face value of debt) as we change the distribution of fundamentals. In Figure 8, we employ two panels – Figure 8a shows the cumulative distribution of liabilities (the endogenous $G(\rho)$ function) and Figure 8b shows the (endogenous) cumulative distribution of prices.

There are two countervailing intuitions at play in this example. First, if we keep $\hat{s}$ fixed, an increase in fundamentals in a FOSD sense leads to lower face values for debt and hence lower endogenous liabilities (this is apparent from Figure 6). The lower liabilities, in turn, lead to higher prices state by state. However, as fundamentals improve, the pool of firms financed at date 0 expands. In particular, the threshold $\hat{s}$ below which firms are financed moves to the right on the x-axis, as can be seen in Figure 6. This means that more levered firms are set up in the economy. If this leverage effect dominates so that with low realizations of fundamentals (θ_2) at date 1, more distress and de-leveraging occur, then market-clearing prices are in fact lower, as is apparent in Figure 7.

In the example discussed above, this second effect dominates, i.e., the pool of firms financed at date 0 is significantly worse. Consequently, an improvement in distribution of fundamentals in a FOSD sense results in worse prices in financial distress. This is consistent with Figure 8a, which shows a higher cumulative distribution (in a FOSD sense) of liabilities when expectations for the future are better. However, we do note that in an ex-ante sense, the probability of reaching these low fundamental states is much lower with better expectation of distribution of θ in a FOSD sense (see Figure 8b which shows the cumulative distribution function of prices $p(\theta)$ under the two distributions). Hence, in expectation prices are still higher, which is precisely why $\hat{s}$ is higher in Figure 6 and higher leverage is sustained at date 0.

This example makes it clear that good times in terms of expectations about credit risk or downside risk enable even poorly capitalized institutions to be funded ex ante and the resulting distribution of leverage in the economy can potentially lead to (il)liquidity effects in prices that are worse during crises that follow better times. Put another way, downside risk or negative skewness of future prices can be higher in good times.

Something like this outcome seems to have accompanied the phenomenon of Great Moderation in developed economies. A sectoral downward shift in volatility over the past two decades appeared to have led to cheap leverage, and thereby gave rise to entry of relatively poorly capitalized institutions in the financial sector. Accompanying this entry was substantial growth in ownership of assets related to residential real estate in these economies. When a severe aggregate shock hit the quality of these assets in the form of housing sector meltdown, de-leveraging and asset sales by highly levered financial institutions ensued. The relatively healthier institutions also possessed little funding liquidity given the deterioration of the real estate assets they held. As a result, asset prices seen seemed far lower than would be expected in the absence of the entry of poorly capitalized institutions.

This counterintuitive phenomenon arises due to the effect of distribution of fundamentals on endogenous entry of firms at date 0. If this entry effect is weak, then prices in our model can be higher state by state at date 1 when the distribution of fundamentals at date 0 is better. To see this possibility, we repeat the example above with a different distribution for borrowing shocks:

$$R(s) = 1 - (1 - \frac{s - 0.2}{t})^{1/\zeta} \ , \tag{35}$$

with $\zeta = 0.05$. In our prior example, the uniform distribution corresponds to $\zeta = 1$. A higher ζ implies lower capital levels and more borrowing at date-0 in a FOSD sense. The distribution with $\zeta = 0.05$ has a much thinner density in the right tail compared to the uniform distribution, reducing the effect of endogenous entry. Figures 9, 10 and 11 show the relevant equilibrium outcomes for this example.

In Figure 9, we see again that $\rho(s)$ is lower when we move to better fundamentals and that $\hat{s}$ is higher. But now in Figure 10, we see that state by state, it is the low fundamentals case ($\gamma = 0.5$) that has the lower price (though the difference is quite small). Here the entry effect, measured as the change in $\hat{s}$, is muted because of the thinness of the left (right) tail in the distribution of initial capital (borrowing) levels.

Figure 11a now shows that the endogenous distribution of liabilities $G(\rho)$ is higher in a FOSD sense for the lower fundamentals case. This explains why prices are lower state by state for weaker fundamentals. Finally, Figure 11b shows that higher fundamentals lead to higher expected prices in an FOSD sense.

We note that we actually found it rather hard to construct this second example in that the right tail of the borrowing distribution had to be thinned considerably. We conjecture that our first example is important and robust. Indeed, it seems reasonable that high expectations lead to more leveraged players being financed, and hence lower prices due to their de-leveraging when really adverse asset-quality states materialize.

4.4 Optimality of debt contracts with lender control

A key aspect of our model is the use of short-term debt contracts which if not rolled over lead to asset liquidations. Alternately, these contracts can be viewed as long-term debt contracts where lenders have interim control rights. In particular, the lender makes a two-period loan but can call the loan at time 1 based on an observable signal of asset quality, inducing the firm to raise external finance or sell assets. This seems to correspond well to the nature of short-term rollover debt such as commercial paper or margins and collateral requirements in financial contracts. We argue in this subsection that in a model of incomplete contracts that follows Aghion and Bolton (1992) (see also Hart and Moore (1994), Hart (1995) and Diamond and Rajan (2001)), the borrowing contract with lender control maximizes the ex-ante financing available to investors.[12]

Our proof consists of two steps. First, we show that debt is the optimal contract. Second, we show that borrower control at date 1 is dominated by lender control at date 1.

Consider any particular realization of asset quality θ_2 at date 1. Suppose for simplicity that accordance of control rights is equivalent to the controlling party making a take-it-or-leave-it offer at date 1. Intuitively, in absence of lender control, the borrower can always invoke the moral hazard problem, that is, threaten to switch to the riskier asset and strategically renegotiate the lender down to $\rho^*(\theta_2)$. This would lower the payoffs to lenders at date 1. In contrast, with lender control, the maximum amount available to lenders by threatening to force asset sales is $p^*(\theta_2) \geq \rho^*(\theta_2)$. Hence, lender control yields higher payoffs to the lender ex post. Ex ante, it is thus in the borrower's interest to give control rights to the lender and raise as much ex-ante debt financing as possible.[13] We formalize this intuition next.

To prove our results, we make two assumptions in the spirit of Aghion and Bolton (1992) and Hart and Moore (1994).

Assumption C1: Courts can verify whether the state 0 occurs or whether $\{y_1, y_2\}$ occurs, however they cannot distinguish between states $\{y_1, y_2\}$.

This assumption essentially states that there is some coarseness in the enforcement ability of courts. While contracts can distinguish between low and high states, they cannot discriminate between different high states.

Assumption C2: While the interim state θ_2 is observable, it is not contractible.

This assumption is similar to Aghion and Bolton (1992) and forces the contract designer to give control conditional on the state θ_2 to either the lender or the borrower. We believe that this assumption is justifiable in the context of financial institutions, especially hedge funds and broker-dealers, as they have complex portfolio strategies with many illiquid positions: the prime broker and hedge fund, for instance, may agree on a valuation, but courts may find it difficult to verify this.

[12] Diamond (2004) in his Presidential address also discusses why short-term debt may resolve incentive problems. He focuses on an environment where the collective action problem makes it hard to renegotiate short-term debt and leads to a run on the firm. This is better for the borrower in an ex-ante sense. Diamond and Rajan (2001) present a similar argument to Diamond (2004).

[13] Note that our model differs from the standard Aghion and Bolton (1992) model in that borrower's ability to invoke the moral hazard problem gives the borrower *too much* power ex post. The only way to limit this is to give the ex-post control rights to the lender.

Assumption C3: Payments at date 1 (ex-post states) cannot be bigger than the maximum payoff in that state or smaller than 0.

This is a standard assumption that limits liability and does not allow payments in excess of what is available.

These three assumptions essentially deliver the result that we want. From Assumption C1, the optimal contract must be a pair $\{0, \rho_i\}$ that pays off the same amount whether states y_1 or y_2 occur (we do not formally prove this).

Assumption C2 implies that we have to compare borrower control or lender control in every state. With borrower control, if $\theta_2\rho_i \leq \rho^*(\theta_2)$, the borrower will honor the contract. However, if $\theta_2\rho_i > \rho^*(\theta_2)$, then the borrower will credibly threaten to switch to the bad project. Hence, the lender will renegotiate the claim from ρ_i to $\frac{\rho^*(\theta_2)}{\theta_2} = f^*(\theta_2)$. Hence with borrower control, the lender gets $\max[\theta_2\rho_i, \rho^*(\theta_2)]$ at date 1.

In contrast, with lender control, the lender can threaten the borrower with liquidation at market prices. Hence, in this case, the lender gets $\max[\theta_2\rho_i, p^*(\theta_2)]$, where $p^*(\theta_2) \geq \rho^*(\theta_2)$ with strict inequality in states with sufficiently high θ_2.

Thus, borrowing with control rights allocated to the lender always generates higher ex-post payoff to the lender and thus greater ex-ante borrowing capacity for the borrower. We state this as a formal result:

Proposition 5 *Under assumptions (C1)–(C3), the optimal contract is debt and lender control always yields a greater region of financed firms than borrower control.*

Proposition 5 provides a rationalization for the structure of financing contracts for trading intermediaries where the moral hazard of risk-shifting is most pertinent. Lenders lend to borrowers and call the loan on interim information unless debt is rolled over. This contract gives strong ex-post control to the lender but reduces the borrower's ability to choose among risky projects and renegotiate. Importantly, in the context of this paper, the Proposition rationalizes the contract structure that we have employed in our preceding analysis and matches the features of margin financing and rollover debt closely.

5 Robustness issues

In this section, we discuss some of the important assumptions that have gone into our analysis and attempt to understand how robust the model is to these assumptions.

5.1 Choice of risk-shifting technology

We acknowledge that our choice of risk-shifting technology from asset 2 to asset 1, as merely switching from a stream of risky to even riskier cash flows without incurring any costs or without engaging in any trades, has the flavor of risk-shifting in the context of real assets. Put another way, in the case of financial assets, one would ideally want the shift of assets to arise because of the sale of risky asset and the purchase of even riskier asset, and potentially clear the markets at date 2 from such shifts. Our choice is based primarily on simplicity and parsimony. Nevertheless, there are at least a few justifications and interpretations that accredit the choice.

First, the shift in assets could represent simply a deterioration in the risk-management function of the financial intermediary, for example, not constraining traders from following doubling-up strategies and allowing (or even encouraging) them to put additional capital at risk so as to "gamble for resurrection." Second, the riskier technology could in fact be outside of the traditional assets invested by the financial sector. Given the risk-shifting incentive, institutions may be willing to pay positive price for the option-value of an asset that otherwise represents a negative net-present value investment. The sellers of such assets from outside of the traditional financial sector may only be too willing to be the recipient of this benefit. An example here would be the "reaching for yield" behavior attributed in recent times to hedge funds, broker-dealers and banks as their alphas or profits from previously successful strategies eroded due to competition. The growth in markets for alternative risks and the "excess" in funding of sub-prime mortgages are again cases in point for the ability of financial institutions to invest in riskier assets at little (ex-ante) cost.

5.2 Specificity in lending and asset markets

A question to raise in our model is why the non-rationed firms do not lend to the rationed firms. One rationale to believe such lending would occur is that players within the financial sector understand each other's assets better and may have superior peer-monitoring technology compared to dispersed or arm's length lenders (such as money market funds who provide commercial paper).[14] Since improved monitoring mitigates the opportunity to engage in asset substitution, such lending would in general improve the funding liquidity of assets. However, equilibrium or no-arbitrage condition between the market for lending and the market for acquiring assets ensures that funding illiquidity persists at least when the moral-hazard intensity is sufficiently severe. The reason for this is that if there is limited funding in the system as a whole, then asset markets will clear only at fire-sale prices, and if this is the case, potential lenders – who are also potential asset acquirers – would be willing to provide financing only at rates that ensure them the same return as the

[14] The assumption of such superior peer-monitoring skills has been employed in the literature to provide a micro-foundation for the existence of inter-bank lending (Rochet and Tirole, 1996).

purchase of cheap assets. Since the face-value of loans would be constrained by the risk-shifting problem, only limited financing would be possible in equilibrium.[15]

The converse of this question is why the financiers in our model (assumed to be of the dispersed type) do not participate in the market for assets. One reason is that such investors, for example money-market funds, are prohibited from investing directly in long-term, illiquid assets (perhaps as a response to their risk-taking incentives). A second reason is that dispersed investors lack the expertise or sophistication to operate complex financial assets.[16] On the one hand, liquidation to such inefficient users would result in allocation inefficiencies in the model as it is more efficient for non-rationed industry insiders to buy all assets from rationed ones. On the other hand, unsophisticated users would not find prices attractive (relative to non-rationed industry insiders) unless fire-sale discount becomes relatively steep. To summarize, as in the original models of Williamson (1988) and Shleifer and Vishny (1992), the idea of asset-specificity is key to ensuring that there is limited participation by financiers in the market for assets. We assume such asset-specificity too.

5.3 Extending date-0 aspects of the model

5.3.1 Insurance arrangements at date 0

In our model, default and de-leveraging at date 1 arise due to asset-side uncertainty coupled with liabilities undertaken by firms at date 0. In principle, such liabilities can be foreseen and hence potentially hedged to an extent by firms through management of asset duration and pre-arrangement of lines of credit (as in Holmstrom and Tirole, 1998). Changing asset duration can be economically expensive. The lines of credit generally contain a Material Adverse Change (MAC) clause, which allows the provider of the line to revoke access in case the borrower's condition has deteriorated sufficiently (see Sufi, 2006 for empirical evidence that this clause is invoked in practice). Indeed, one reason why such clauses might feature in optimal contracting of the line of credit is precisely to avoid agency problems tied to borrower-lender relationships. In

[15] See Acharya, Shin and Yorulmazer (2007) for modeling of such linkages between markets for financial and real assets. Further, Acharya, Gromb and Yorulmazer (2007) argue that in cases where a large number of players are liquidity takers and only a handful remain as potential liquidity providers, the providers may act strategically and charge higher than competitive lending rates in order to force greater asset sales and extract further price discounts. Similar arguments based on strategic motives have also been made in the context of predatory trading in capital markets by Brunnermeier and Pedersen (2005b) and Carlin, Lobo and Viswanathan (2007).

[16] This has been witnessed painfully during the sub-prime collapse of Summer 2007. The opacity of balance-sheets of financial institutions and the inability of even sophisticated lenders such as prime brokers to value complex products like CDO and CLO tranches (and the lack of any secondary trading platform for the same) seem to have led to a freeze in inter-bank lending, securitization and financing of assets such as leveraged buyouts that rely on such securitization.

our view, it is thus natural to model asset sales and de-leveraging as the result of some residual asset-liability mis-match on the balance-sheets of financial firms.[17]

5.3.2 Risk-shifting and buffering liquidity at date 0

Our model considers risk-shifting after new debt has been issued (or initial debt has been rolled over) at date 1. One key question is whether firms could engage in risk-shifting at date 0 or conversely whether firms could save cash at date 0.

We assume that a switch to the risk-shifting technology is irreversible. Recall that the risk-shifting technology is specific to each firm. Then, the claim is that a firm will not risk-shift at date 0. Intuitively, the option to risk-shift is worth more alive than dead, i.e., early exercise of this option is never optimal. More formally, whether a firm is liquidated at date 1 or not is only a function of its leverage and realization of asset quality θ_2. If θ_2 is realized as low enough relative to the firm's leverage, creditors will attempt to liquidate, and if the firm has already switched to the riskier asset, then liquidation value is zero as there are no alternative buyers. In these states, firm owners make no return. If θ_2 is high enough relative to leverage, then creditors do not liquidate the firm. In these states, the firm has the option to switch to the riskier technology if it is optimal to do so and provided it did not risk-shift already at date 0. However, if θ_2 is high enough, the switch to the riskier asset is not necessarily desirable. Had the firm already risk-shifted at date 0, it would give up this option of choosing its risk at date 1 after asset quality θ_2 is realized. This is because the firm would be unable to switch back to the better asset, implying that it is never optimal to risk-shift in our model at date 0.

Similarly, a firm that invests at date 0 will never hold any excess cash. This follows from the convexity of the $\rho(s)$ function. Hoarding one unit of cash increases the shortfall s by one dollar, which leads to an increase in date-1 liability $\rho(s)$ of say $x \geq 1$. At date 1, the firm has an extra dollar of cash but more than an extra dollar of liability. If the firm's liability turns out to be bigger than $\rho^*(\theta_2)$, it needs to raise an extra $(x-1)$ units of funding. If its liability turns out to be lower than $\rho^*(\theta_2)$, then it loses debt capacity equivalent to $(x-1)$ units and hence buys less assets from distressed firms. In either case, it is suboptimal to have borrowed and held cash.

[17] Especially in our context of financial intermediaries, such mis-match can be more broadly interpreted as arising due to change in the mark-to-market valuations of financial derivatives such as swaps where the ex-ante contract values are zero, but ex post, depending upon the realization of underlying risks, the valuation may transform the position into an asset or a liability. Such risks are generally not hedged perfectly as that would be tantamount to completely undoing the position undertaken through the security in the first place.

5.3.3 Market-clearing at date 0

One limitation of date-0 aspect of our model is that if a financial firm can raise financing to meet its shortfall, it enters the intermediation sector and is essentially "endowed" with the asset or the technology. That is, we do not have a market for this asset at date 0 where endogenous entry of financial firms, coupled perhaps with an exogenous supply, determines the price at which firms acquire the asset. It would certainly be interesting to examine such market-clearing. Qualitatively, we conjecture that our primary insight that better fundamentals lead to greater entry at date 0 and lower prices at date 1 would survive, but in a slightly different guise. With market-clearing at date 0, entry would raise the price of the asset at date 0 when fundamentals are better, dampening the effect of entry to an extent, but when prices at date 1 are examined *relative* to the price at date 0, a similar conclusion would arise. That is, better fundamentals would lead to lower prices at date 1 relative to the price at date 0, and thus give rise to higher inter-temporal volatility in prices. While we interpret lowering of date-1 prices as more severe crises in the current model, the model with market-clearing at date 0 could be interpreted as leading to a "bubble" at date 0 (in the spirit of Lorenzoni (2007)'s "credit boom").

5.3.4 Entry of date-0 rationed borrowers at date 1

Our model does not allow original investors to choose between investing and waiting. Rampini and Viswanathan (2007) analyze these tradeoffs in a different model with walk-away constraints and full contingent claims (but exogenous capital prices). In their model, agents with low productivity choose to wait; in our model, this would translate into agents with low wealth. In principle, the staying out of low-capitalized agents at date 0 would weaken the endogenous entry effect that in our numerical examples led to lower prices with better fundamentals. One could argue, however, that there may be learning-by-doing effects so that the better-capitalized insiders may stand a relative advantage to the poorly-capitalized outsiders, except in the extreme situation where almost all insiders are in distress. Acharya, Shin and Yorulmazer (2009) analyze such a model with difference in expertise between insiders and outsiders, and how this affects the ex-ante equilibrium choice to be an insider or outsider. Neither of these papers, however, considers the endogenous pricing and quantity of leverage and attendant agency problems.

6 Conclusion

In this paper, we presented a moral-hazard based, agency-theoretic to study the important role played by leverage in generating asset fire sales and the resulting asset-pricing implications. We reiterate that our model of the Shleifer and Vishny (1992)'s industry-equilibrium argument for debt capacity is characterized entirely by a single parameter, the moral-hazard intensity, which drives the funding liquidity of assets, the extent of de-leveraging, and the level of equilibrium prices. This characterization is crucial to understanding the fundamental link between funding and market illiquidity witnessed during financial crises. Our most surprising result was the phenomenon that economies with lower volatility or better fundamentals are associated with greater entry of highly levered financial institutions, so that when adverse asset shocks materialize such economies experience greater asset-price deterioration.

In ongoing work, we are examining the possibility of contagion across asset markets, when there is uncertainty about portfolio composition of financial institutions and there is a shock in fundamentals to some of the assets. Such uncertainty, resulting from the opaqueness of increasingly complex balance-sheets of trading institutions (and to an extent, necessary for them to prevent erosion of their "alphas"), and coupled with de-leveraging, has been argued to be a significant contributor in the sub-prime crisis to market and funding liquidity problems. The spillover of the collapse of the market for mortgage-backed assets onto broader credit markets presents a case in point.

We believe that such pursuits represent only the tip of the iceberg and much work remains in integrating agency-theoretic corporate-finance issues into main-stream asset-pricing literature, especially in the context of understanding liquidity issues in truly dynamic set-ups. The simple building block of this paper, based on an agency problem central to financial institutions – namely, leverage-induced risk-shifting or asset-substitution – may serve as a useful starting point for such modeling.

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Appendix 1

Proof of Proposition 2: We first prove that the market-clearing price p^* exists and is unique.

Step 1. The demand function for assets is given by

$$D(p,\rho^*) = \begin{cases} \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^*-\rho)}{(p-\rho^*)}\, g(\rho)d\rho & \text{if } \rho^* \leq p \\ \left[0, \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^*-\rho)}{(p-\rho^*)}\, g(\rho)d\rho\right] & \text{if } p = \overline{p} \end{cases}$$

where at price $\overline{p}$, we get an interval of possible demand as buyers are indifferent between not buying and buying up to their maximum liquidity. Hence, the excess demand function is given by

$$E(p,\rho^*) = \begin{cases} -1 + \frac{1}{(p-\rho^*)} \int_{\rho_{\min}}^{p} G(\rho)d\rho & \text{if } \rho^* \leq p \\ \left[-1 + \frac{1}{(p-\rho^*)} \int_{\rho_{\min}}^{p} G(\rho)d\rho, -\int_{\rho^*}^{p} \frac{(\rho-\rho^*)}{(p-\rho^*)}\, g(\rho)d\rho - \int_{p}^{\rho_{\max}} g(\rho)d\rho\right] & \text{if } p = \overline{p} \end{cases}$$

where as before we get an interval at $\overline{p}$.

Step 2. Note that the excess demand for $p = \rho^*$ is positive infinity.

Step 3. If the excess demand is positive for all $p < \overline{p}$, the price must be $\overline{p}$ as at $\overline{p}$ the interval definition of excess demand above includes 0. So, $\overline{p}$ is the only feasible price. Intuitively, if there are more agents willing to buy than sell at the highest possible price, this must be the price.

Step 4. If the excess demand is negative as $p \to \overline{p}$, we must have at least one solution for p. However, we note that for $\rho^* < p < \overline{p}$, the derivative of the excess demand (when the excess demand is ≥ 0) is given by

$$\frac{\partial E(p, \rho^*)}{\partial p} = \quad -\tfrac{1}{(p-\rho^*)^2} \int_{\rho_{\min}}^{p} G(\rho) d\rho + \tfrac{G(p)}{p-\rho^*} \quad \leq -\tfrac{1}{(p-\rho^*)} + \tfrac{G(p)}{p-\rho^*} \quad < 0, \tag{36}$$

where we have used the fact that a positive excess demand implies that $\frac{1}{(p-\rho^*)} \int_{\rho_{\min}}^{p} G(\rho) d\rho \geq 1$ and that $G(p) < 1$.

Hence when the excess demand is zero, its derivative must also be negative, thus we can only have one price that sets the excess demand to zero and the price $\overline{p}$ is unique.

Step 5. To prove that p^* is increasing in ρ^*, note that the excess demand function has a positive derivative with respect to ρ^* for all $p < \overline{p}$ (as can be verified using the expression for excess demand in Step 1 above). Since the excess demand function is strictly downward sloping for positive excess demand, it immediately follows that p^* is strictly increasing in ρ^* if $p^* < \overline{p}$; otherwise the price just stays at $\overline{p}$.

Step 6. It follows from Step 5 that there exists a unique critical value $\hat{\rho}^* \in (\rho_{\min}, \overline{p})$ such that the market-clearing price $p^* = \overline{p}, \forall \rho^* \geq \hat{\rho}^*$ and $p^* < \overline{p}$ otherwise, in which case p^* satisfies equation (14). Note also from equation (14) that we must have $p^* \geq \rho^*$ with equality arising only when $\rho^* = \rho_{\min}$.

This completes the proof. $\diamondsuit$

Completion of Proof of Proposition 4:

We now fill in the details of the contraction mapping theorem that we use to prove existence and uniqueness. Granas and Dugundji (2003), Theorem 2.1, shows a general approach to existence of Volterra integral equations of the second kind, we adapt their proof to our set up.

We first show that if $f(\rho, t)$ is Lipschitz in t with Lipschitz constant L_1 and $G(\rho)$ is Lipschitz in ρ with Lipschitz constant L_2, we can prove existence and uniqueness, at the end of the proof we provide sufficient conditions of the Lipschitz continuity of these functions.

Let $L = \max\{L_1, \frac{L_2}{R(\hat{s})}\}$

Let $\mathbf{E}$ be the Banach space of all continuous real valued function on $[\theta_1 y_1, \theta_{\max} y_2]$ equipped with the norm

$$||s|| = \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} |s(\rho)| \tag{37}$$

This norm is equivalent to the standard sup norm $||x||_s$ (a function Lipschitzian in one norm is

Lipschitzian in any equivalent norm) because

$$e^{-L\theta_{max}y_2}||x||_s \leq ||x|| \leq ||x||_s, \tag{38}$$

further it is complete.

Define $M(s)(\rho) = \int_{\theta_1 y_1}^{\rho} \frac{R(s(u))}{R(\hat{s})} du$ where s refers to the function $s(\rho)$ on $[\theta_1 y_1, \theta_{\max} y_2]$. We first note that

$$\begin{aligned}
& ||M(s') - M(s)|| \\
\leq \; & \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |\frac{R(s'(u)}{R(\hat{s})} - \frac{R(s(u)}{R(\hat{s})}| du \\
\leq \; & \frac{L_1}{R(\hat{s})} \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |s'(u) - s(u)| du \\
\leq \; & L \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |s'(u) - s(u)| du \\
\leq \; & L \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} e^{Lu} e^{-Lu} |s'(u) - s(u)| du \\
\leq \; & L||s' - s|| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} e^{Lu} du \\
= \; & L||s' - s|| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \frac{e^{L\rho} - e^{L\theta_1 y_1}}{L} \\
\leq \; & (1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)}) ||s' - s||
\end{aligned} \tag{39}$$

Next define the map F:$\mathbf{E} \rightarrow \mathbf{E}$ by

$$F(s)(\rho) = \int_{\theta_1 y_1} \rho f(t, M(s)(t)) d\rho \tag{40}$$

where s is the function $s(\rho)$. We wish to show this is a contractive map, hence

$$\begin{aligned}
& ||F(s') - F(s)|| \\
\leq \; & \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |f(t, M(s')(t) - f(t, M(s)(t)| dt \\
\leq \; & L \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} |M(s')(t) - M(s)(t)| dt \\
\leq \; & L||M(s') - M(s)|| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \int_{\theta_1 y_1}^{\rho} e^{Lt} dt \\
\leq \; & L||M(s') - M(s)|| \max_{\theta_1 y_1 \leq \rho \leq \theta_{\max} y_2} e^{-L\rho} \frac{e^{L\rho} - e^{L\theta_1 y_1}}{L}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad (1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)}) \quad ||M(s') - M(s)|| \\
&\leq \quad (1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)})^2 \quad ||s' - s|| \qquad (41)
\end{aligned}
$$

which is contractive as $(1 - e^{-L(\theta_{\max} y_2 - \theta_1 y_1)}) < 1$. Hence by the Banach contraction theorem, we have a unique fixed point in $\mathbf{E}$ and the sequence given by successive iterations $F^n(s)$ converges to this unique fixed point uniformly in the norm $|| \cdot ||$ and hence in the standard sup norm $|| \cdot ||_s$.

We now fill in the details of Lipschitz continuity. We know that if f is differentiable with bounded derivative $f'(\rho) \leq L$, the f is Lipschitz with constant $K < L$. It suffices for the cdf R to assume that it has bounded derivative over the interval $[\theta_1 y_1, \theta_{\max} y_2]$.

To prove that the function $f(\rho, t)$ defined by

$$
f(\rho, t) = 1 - H\left(\max\left\{\frac{(\theta_1 y_1 + (\rho - t) + \sqrt{(\theta_1 y_1 + (\rho - t))^2 - 4 y_2 (\rho - t)\theta_1}}{2 y_2}, \frac{\rho}{y_2}\right\}\right) \qquad (42)
$$

is Lipschitz in ρ, define the auxiliary function

$$
\hat{f}(\rho, t) = 1 - H\left(\frac{(\theta_1 y_1 + (\rho - t) + \sqrt{(\theta_1 y_1 + (\rho - t))^2 - 4 y_2 (\rho - t)\theta_1}}{2 y_2}\right) \qquad (43)
$$

which is Lipschitz is ρ provided the cdf $H(\cdot)$ is differentiable with bounded derivatives. But this suffices for function f. Given ρ, let $\hat{t}(\rho)$ be the point where the two terms in the maximum function are equal (the function is not differentiable at this point in t). If $t, t' \geq \hat{t}(\rho)$, the Lipschitz continuity of $\hat{f}(\rho, t)$ in t suffices. If $t > \hat{t}(\rho) > t'$ we note that $|f(\rho, t| - f(\rho, t')| = |f(\rho, t) - f(\rho, \hat{t}(\rho))|$ and we can use the Lipschitz continuity of $\hat{f}(\rho, t)$ in t as follows:

$$
\begin{aligned}
& \quad f(\rho, t) - f(\rho, t') \\
&= \quad f(\rho, t) - f(\rho, \hat{t}(\rho)) \\
&= \quad \hat{f}(\rho, t) - \hat{f}(\rho, \hat{t}(\rho)) \\
&= \quad L_1 |\hat{t}(\rho) - t| \\
&\leq \quad L_1 |t' - t| \qquad (44)
\end{aligned}
$$

which completes the proof of Lipschitz continuity of the function $f(\rho, t)$ in t. $\diamondsuit$

Appendix 2

Solving the integro-differential equation

We now discuss the numerical method used to solve the integro-differential equation,

$$\frac{ds}{d\rho} = 1 - H\left(\max\left\{\frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2}\right\}\right) \tag{45}$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$.

Find the initial value of $\hat{s}$ on $[\theta_1 y_1, \theta_{\max} y_2]$ as follows,

$$\hat{s} = \int_{\theta_{\min}}^{\theta_{\max}} \theta_2 y_2 h(\theta_2) d\theta_2 \tag{46}$$

where we have used the fact that $\theta_2 y_2$ is the highest possible price in each state.

The recursive algorithm works as follows. Start with $s(\rho) = \rho$ on $[\theta_1 y_1, \theta_{\max} y_2]$. Use this to derive a first order Riemann sum numerical approximation to the integral on a discrete grid $[t_0 = \theta_1 y_1, t_1, \ldots, t_N = \theta_{\max} y_2]$ as

$$\int_{\theta_1 y_1}^{t_n} \hat{G}(\rho) d\rho = \sum_{k=1}^{n} (t_k - t_{k-1}) \hat{G}(t_{k-1}). \tag{47}$$

For each t_n,

$$L(t_n) = t_n - \sum_{k=1}^{n} (t_k - t_{k-1}) \hat{G}(t_{k-1}). \tag{48}$$

The integro-differential equation is then approximated by the first order Taylor expansion

$$\begin{aligned} & s(t_{n+1}) \\ & \quad = \quad s_n + (t_{n+1} - t_n)\left(1 - H\left(\max\left\{\frac{(\theta_1 y_1 + L(t_n)) + \sqrt{(\theta_1 y_1 + L(t_n))^2 - 4y_2 L(t_n)\theta_1}}{2y_2}, \frac{t_n}{y_2}\right\}\right)\right) \end{aligned} \tag{49}$$

This yields a new grid approximation $s(t_n)$, we set the value of $\hat{s}$ as $s(t_N)$. Now repeat the above process until convergence occurs (maximum difference in $s(t_n)$ is 0.001). This ensures that $\hat{s}$ also converges.

t = 1			t = 2
LIQUIDITY SHOCKS	**DEBT FINANCING**	**MARKET FOR ASSETS AT PRICE p**	**MORAL HAZARD PROBLEM**
	• Firms with low ρ_i : - Borrow to rollover debt and potentially buy assets	- Buy assets	• Choose between safe and risky asset • Assets pay off, debt is due
• Firm i has liability outstanding of ρ_i	• Firms with moderate ρ_i : - Credit-rationed - Borrow to raise $(\rho_i - \alpha_i p)$	- "De-lever" by liquidating α_i assets	• Choose between safe and risky asset • Assets pay off, debt is due
	• Firms with high ρ_i : - Credit-rationed	- Are entirely liquidated	

Figure 1: Timeline of the benchmark model.

Figure 2: Equilibrium price p* as a function of (inverse) moral hazard intensity

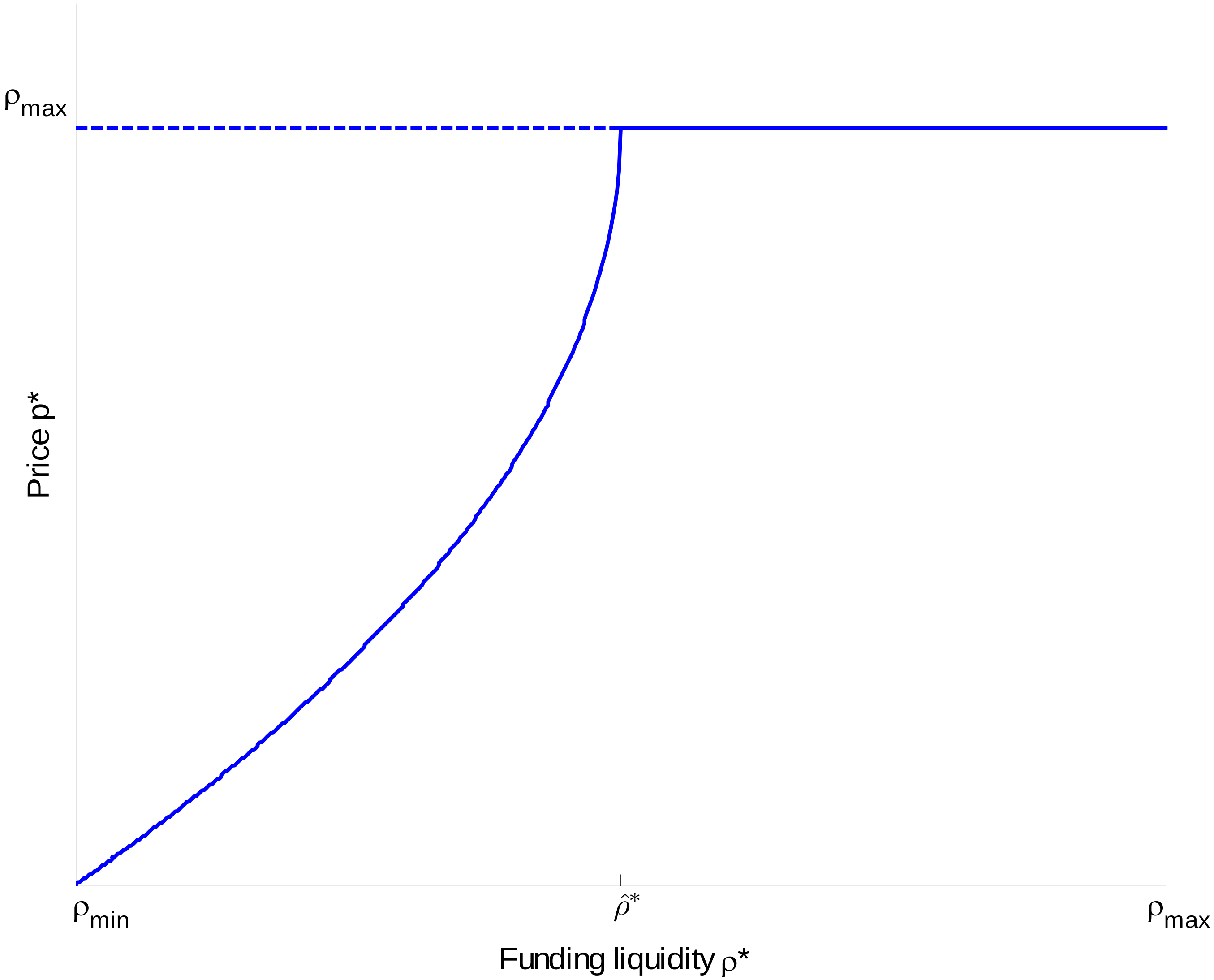


Figure 3: Equilibrium de-leveraging or asset-sale proceeds as a function of leverage ρ

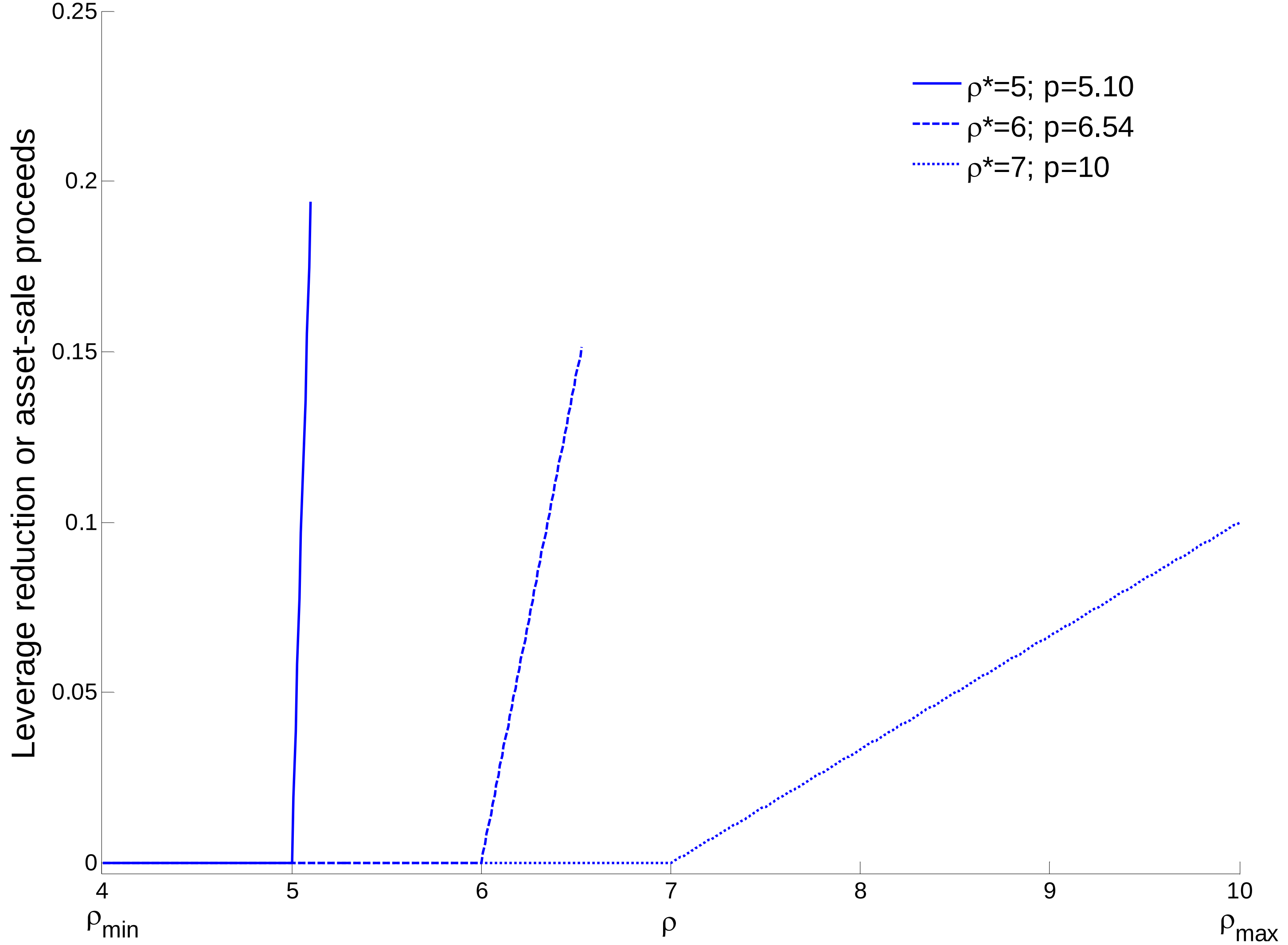
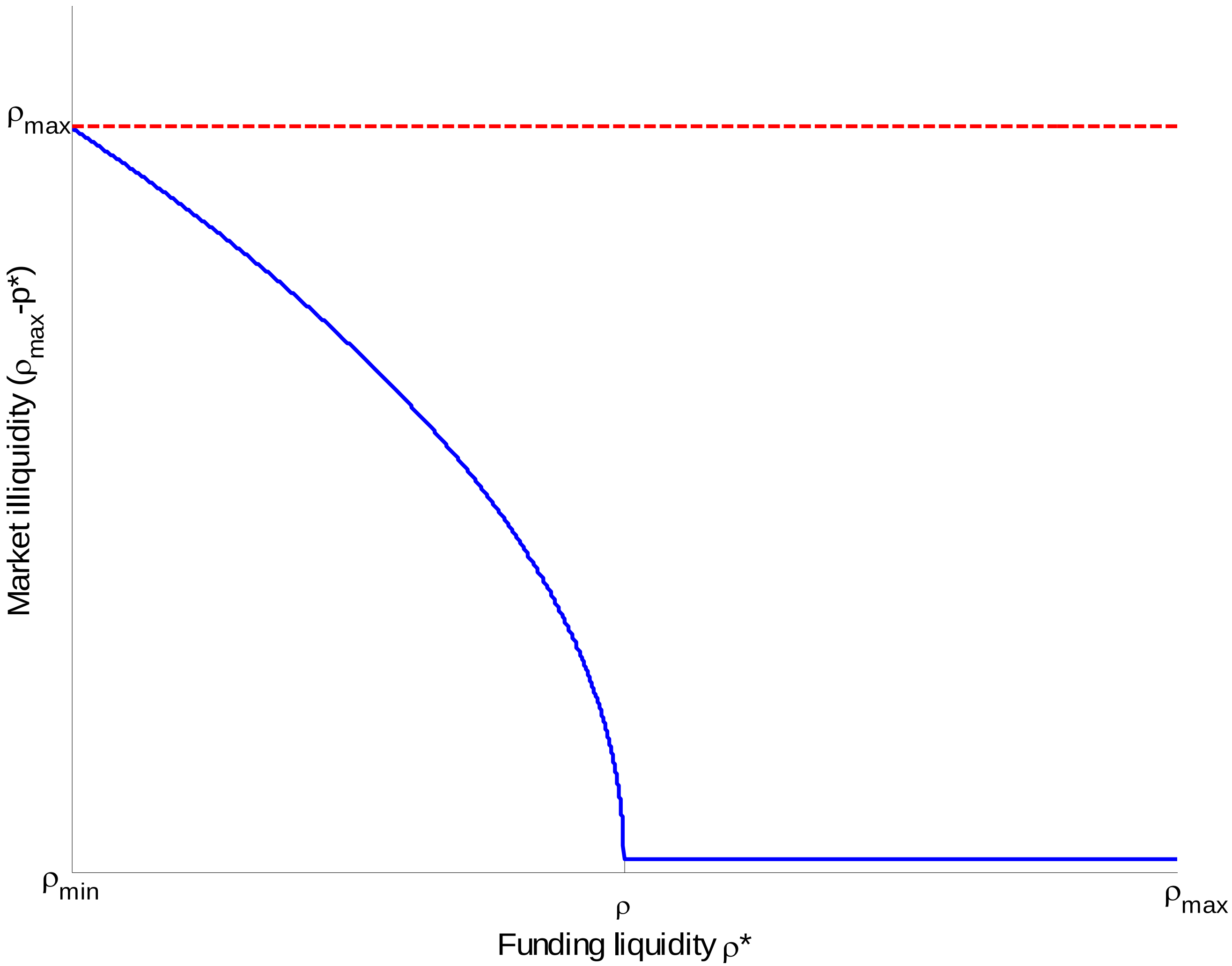


Figure 4: The relationship between market (il)liquidity and funding liquidity



t = 0	t = 1		t = 2
FIRST ROUND OF BORROWING	**DEBT DUE** **ADDITIONAL DEBT FINANCING** **REALIZATION OF ASSET QUALITY θ_2**	**MARKET FOR ASSETS** **AT PRICE $p(\theta_2)$**	**MORAL HAZARD PROBLEM**
	• Firms with low ρ_i : - Borrow to rollover debt and potentially buy assets	- Buy assets	• Choose between safe and risky asset • Assets pay off, debt is due
• Firm i with sufficiently high wealth w_i raises $(I - w_i)$ with debt of face value ρ_i ; Firms with very low wealth are rationed	• Firms with moderate ρ_i : - Credit-rationed - Borrow to raise $(\rho_i - \alpha_i p)$	- "De-lever" by liquidating α_i assets	• Choose between safe and risky asset • Assets pay off, debt is due
	• Firms with high ρ_i : - Credit-rationed	- Are entirely liquidated	

Figure 5: Timeline of the augmented model.

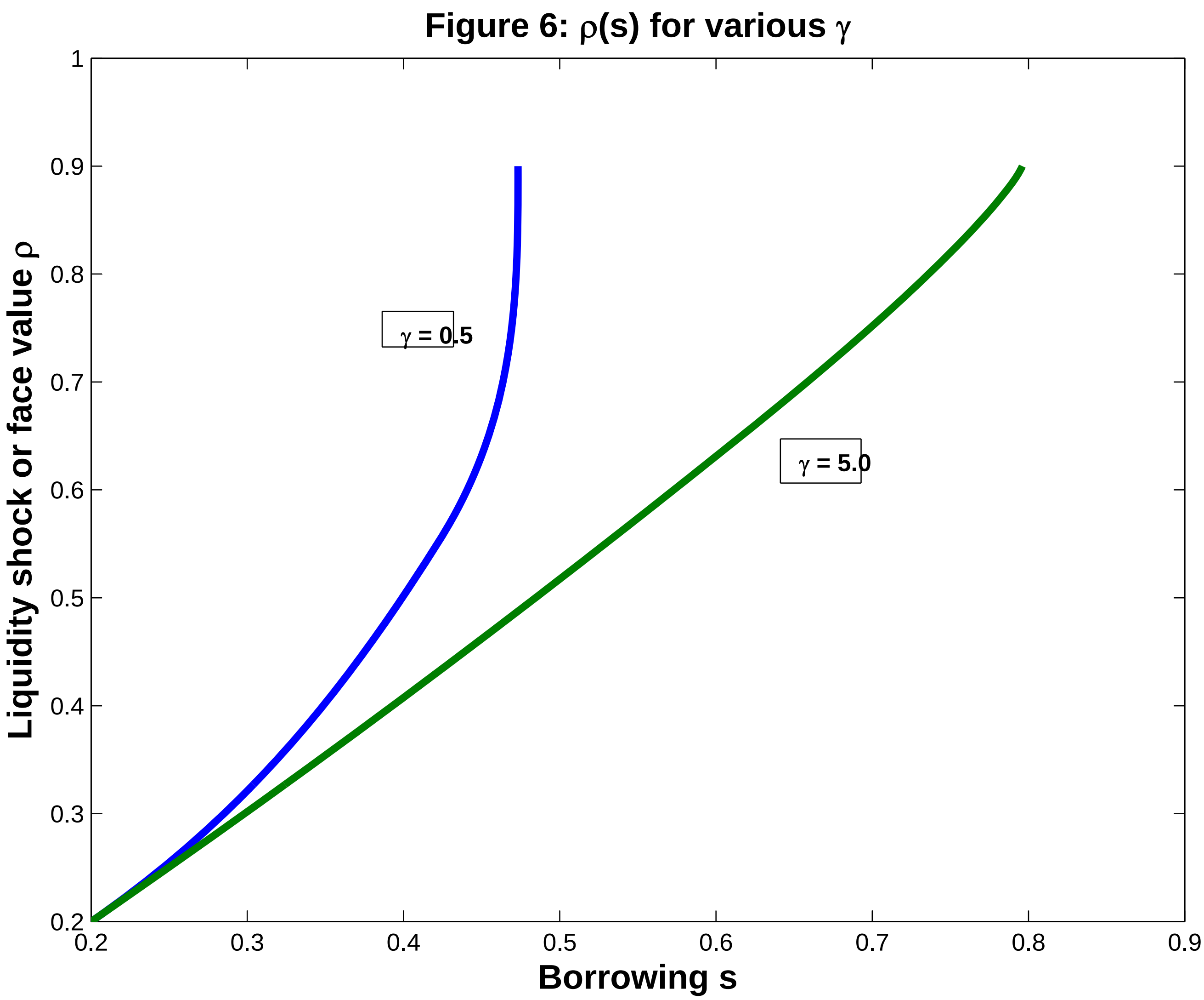
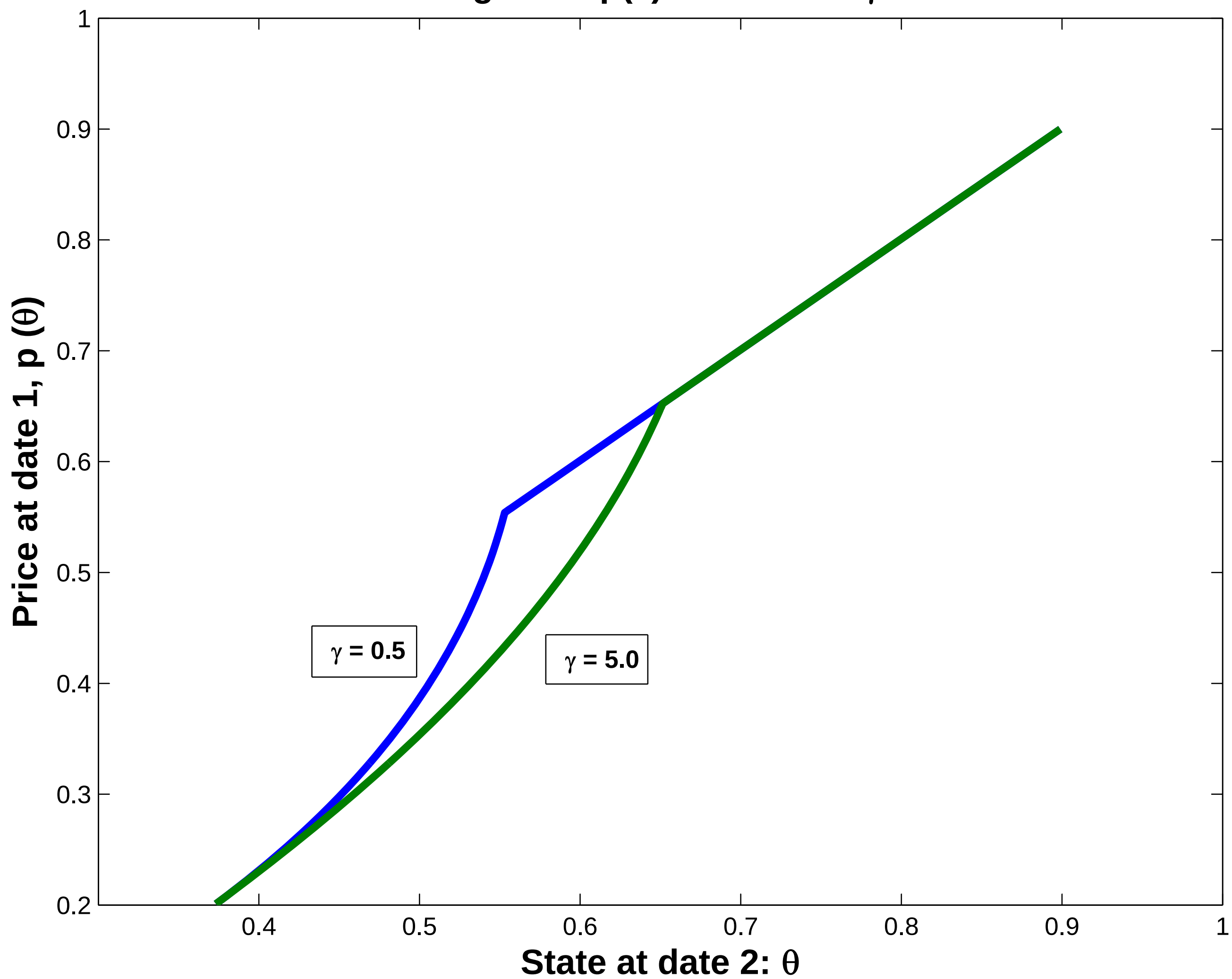


Figure 6: ρ(s) for various γ
Liquidity shock or face value ρ
Borrowing s
γ = 0.5
γ = 5.0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

Figure 7: p(θ) for various γ



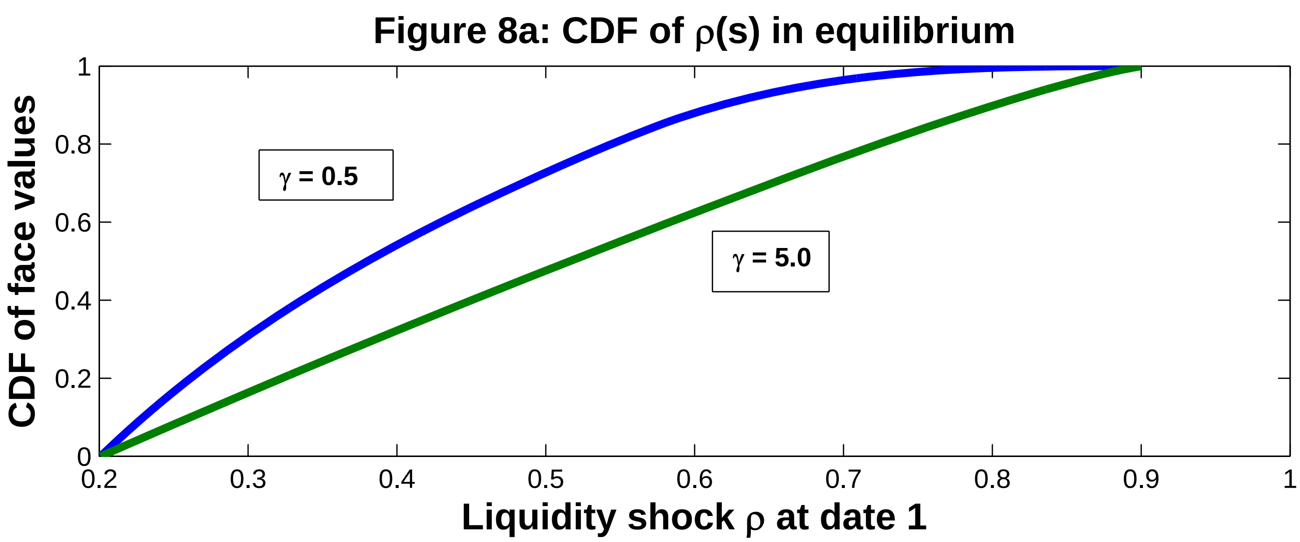


Figure 8a: CDF of ρ(s) in equilibrium
CDF of face values
γ = 0.5
γ = 5.0
0
0.2
0.4
0.6
0.8
1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Liquidity shock ρ at date 1

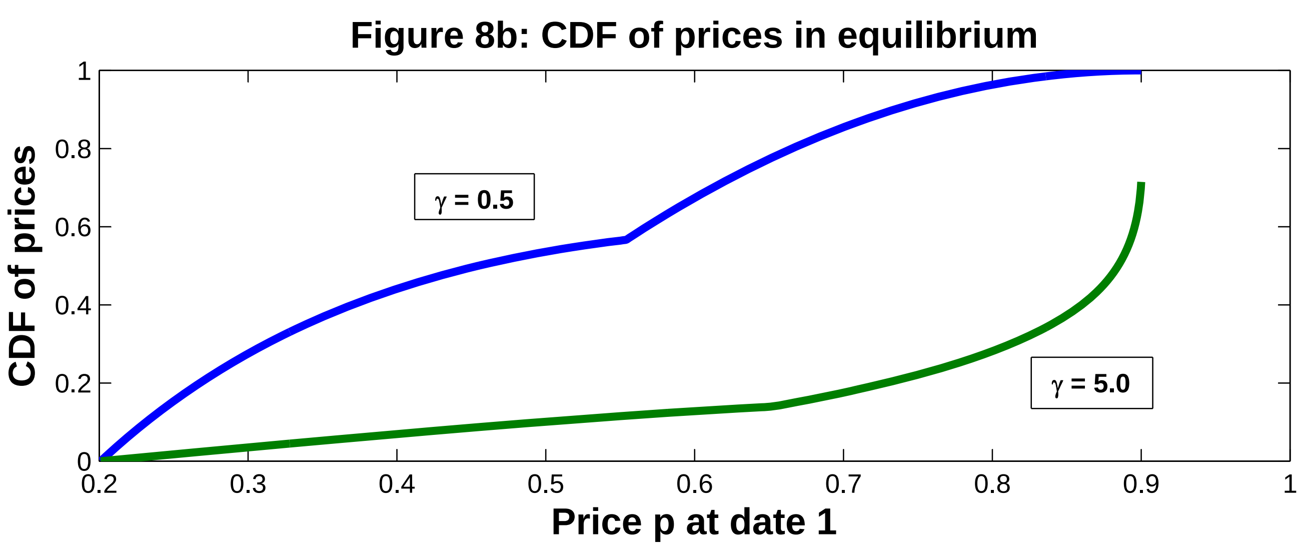


Figure 8b: CDF of prices in equilibrium
CDF of prices
γ = 0.5
γ = 5.0
0
0.2
0.4
0.6
0.8
1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Price p at date 1

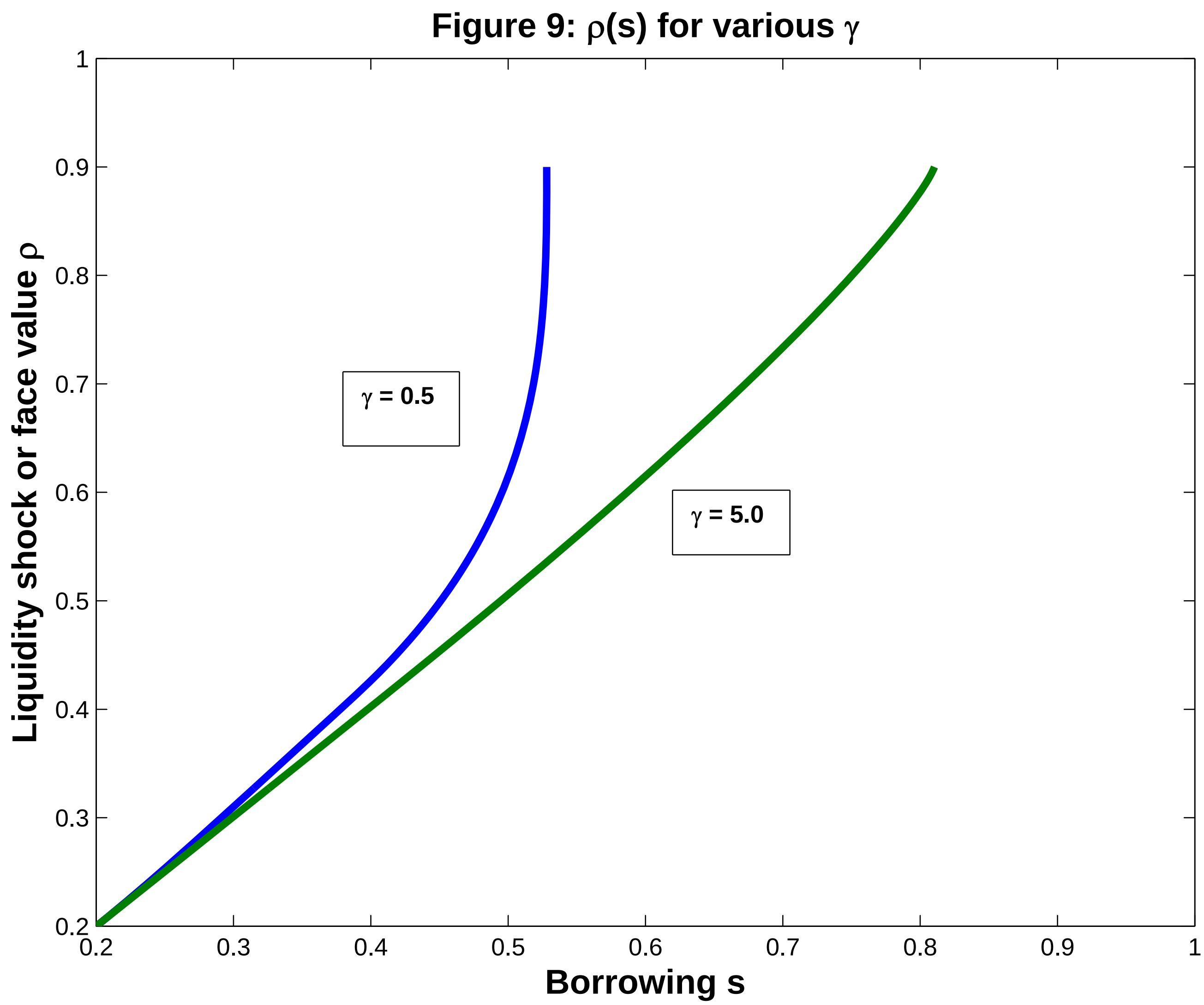


Figure 9: ρ(s) for various γ
Liquidity shock or face value ρ
Borrowing s
γ = 0.5
γ = 5.0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

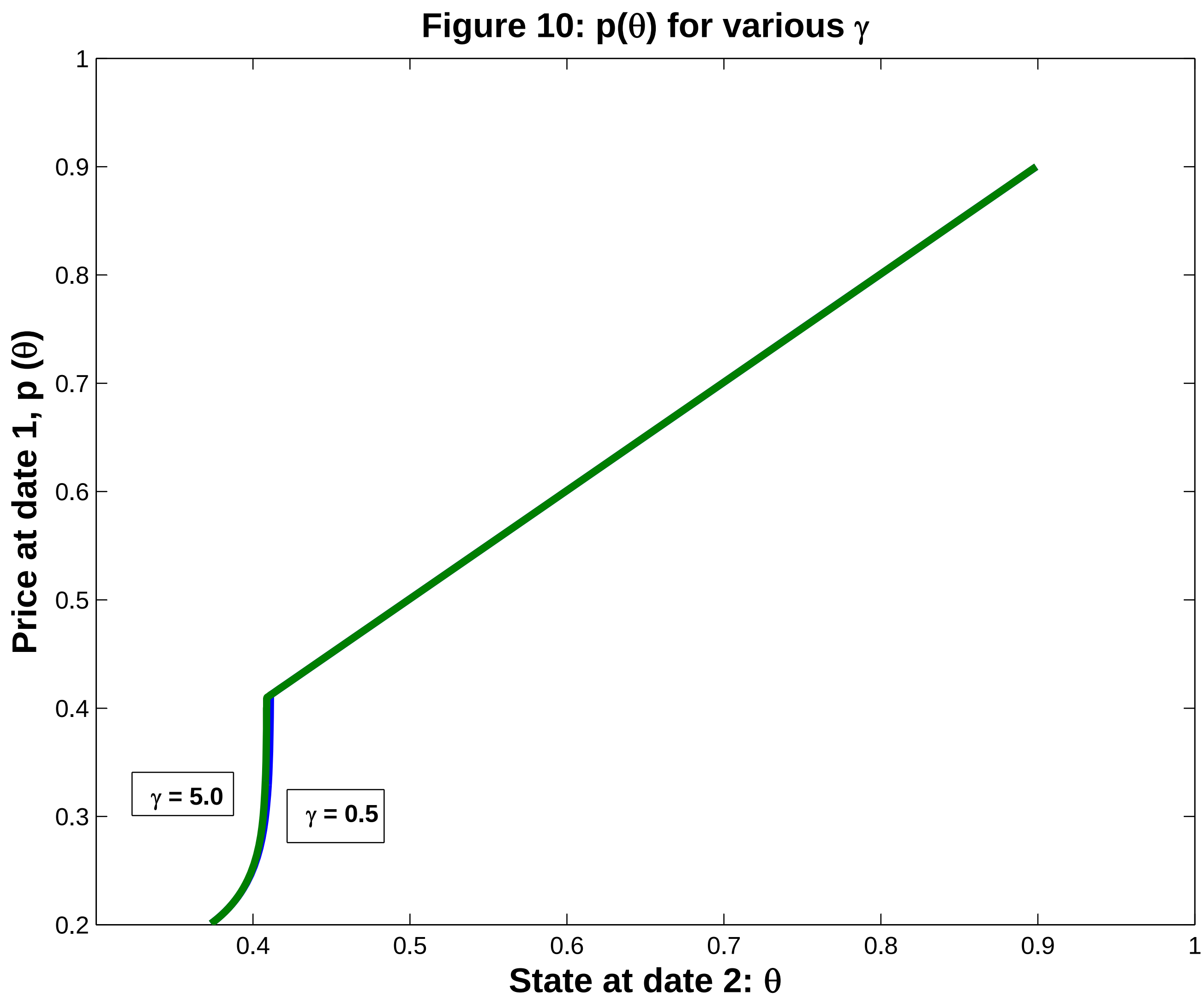


Figure 10: p(θ) for various γ
Price at date 1, p (θ)
State at date 2: θ
γ = 5.0
γ = 0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.4
0.5
0.6
0.7
0.8
0.9
1

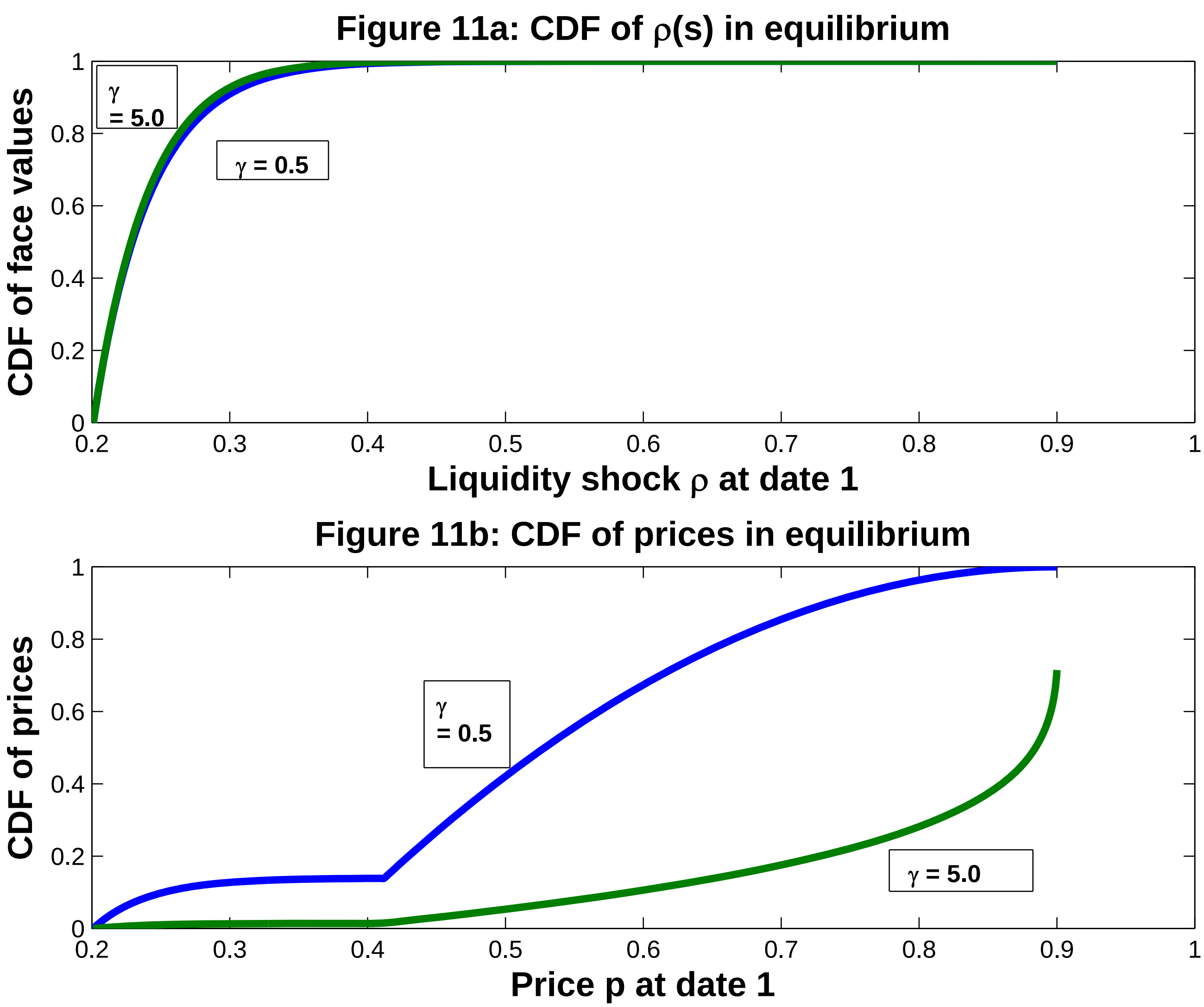


Figure 11a: CDF of ρ(s) in equilibrium
CDF of face values
Liquidity shock ρ at date 1
γ = 5.0
γ = 0.5
Figure 11b: CDF of prices in equilibrium
CDF of prices
Price p at date 1
γ = 0.5
γ = 5.0