Central Bank Haircut Policy*

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Abstract

The paper develops a model to study the optimal choice of the central bank haircut policy. In the presence of uncertainties regrading liquidity needs and asset prices, there is a trade-off between providing liquidity to constrained agents and controlling the abundance of liquidity in the economy. The choice of the haircut involves balancing impacts on the liquidity positions of agents with different portfolio choices and different liquidity needs. In general, a full haircut and a zero haircut are both sub-optimal. The optimal choice will depend on the relative tightness of agents' liquidity constraints, the predictability of the liquidity shocks, and the volatility of asset prices. The optimal haircut is higher when the central bank is unable to lend exclusively to agents who actually need liquidity. Finally, for a temporary, surprise drop in the haircut, the central bank can be more aggressive than setting the permanent level of the haircut.

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1 Introduction

The comparative advantage of a central bank is in the production and dissemination of highly liquid assets that aid in trade. In this paper we examine who a central bank should

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make loans of a liquid asset (in this case money) that are collateralized by illiquid and risky assets. This is an important question since in an economy with decentralized trading and idiosyncratic liquidity shocks, the equilibrium allocation is typically inefficient because some agents are liquidity constrained due to their ex-post excessive holding of illiquid assets.

In this situation, a benevolent central bank may desire to provide liquidity to constrained agents by using a lending facility. In the lack of perfect enforcement of these loans, the illiquid assets can be taken as collateral for borrowing the central bank loans. However, the value of this collateral can change over time, and, thus, it is necessary to require a pledge of collateral large enough to adequately cover losses in the event of a default. The magnitude by which the initial value of the collateral is discounted is called the "haircut".

One may argue that it is *ex-post* efficient to impose a low haircut in order to effectively insure agents against their idiosyncratic liquidity shocks. What is ignored is that, as long as the central bank is not setting a full haircut, the lending facility provides not only insurance against liquidity shocks, but also provides insurance against asset price declines. Thus, by changing the haircut, the agents' portfolio choice, their incentive to default and the returns to different assets will be changed endogenously.

The above discussion leads to the central question we seek to address; when a central bank provides liquidity to financial institutions by collateralized loans, how should it design its collateral policy? In particular, how should it determine its haircut policy? This paper aims to develop a model to study the design of the haircut policy in the Canadian Large Value Payment System (LVTS); although our results are general enough to cover other types of collateralized central bank liquidity facilities.

To help answer these question of how a central bank should design its collateral and haircut policy we build a model that has four key features: A portfolio choice between liquid and illiquid assets, idiosyncratic liquidity shocks, strategic default, and asset price uncertainty.

First, agents make a portfolio choice between liquid and illiquid assets. Second, idiosyncratic liquidity shocks are realized after they have made their portfolio choice. Therefore, some agents may end up holding too much illiquid assets when they are facing a high liquidity need. In the absence of an intra-day money market, there is a role for the central bank lending facility. Third, these loans are subject to potential default by the borrowers. This motivates the need to require borrowers to pledge collateral when receiving a loan from the central bank. Forth, the asset price of the collateral is uncertain at the time the loan is made. This generates the need to impose a haircut on the collateral. We will use this model to study how changing the haircut policy will induce the endogenous response of default and portfolio choice. When the central bank does not possess the fiscal power to collect revenue from agents through non-distortionary tax instruments, there exists a trade-off between liquidity insurance and the distortion generated by decreasing the haircut.

We show that lowering the haircut will have different effects on agents depending on both their portfolios and different liquidity needs. On the one hand, it can relax the liquidity constraint of *illiquid asset holders*. On the other hand, it will lower the value of liquid assets (e.g. money) by both reducing the returns to holding liquidity and increasing the cost of holding liquidity. Therefore, it will tighten the liquidity constraint of *liquid asset holders*. As a result, there is a trade-off between lowering the haircut and lowering the interest cost of holding liquidity. In general, in the absence of instruments to freely withdraw extra liquidity from the economy, it is not feasible for policy-maker to lower the haircut and the interest rate simultaneously. The choice of the haircut involves balancing impacts on the liquidity positions of different groups.

The optimal choice will depend on the relative tightness of agents' liquidity constraints, the sizes of different groups (which depends on the predictability of the liquidity shocks), and the volatility of asset prices. In general, a full haircut and a zero haircut are both sub-optimal.

We also point out that one key factor is whether the central bank is able to lend exclusively to agents who actually need liquidity. When exclusive lending is not feasible, the cost of providing liquidity insurance to the illiquid asset holders by lowering the haircut becomes more costly in terms of distorting the liquid asset holders' liquidity constraint. Owing to this trade-off, it is generally not optimal to set the haircut too low.

Finally, we also illustrate that, if the central bank can commit not to repeat in the future, a temporary, surprise cut in the haircut can be welfare improving.

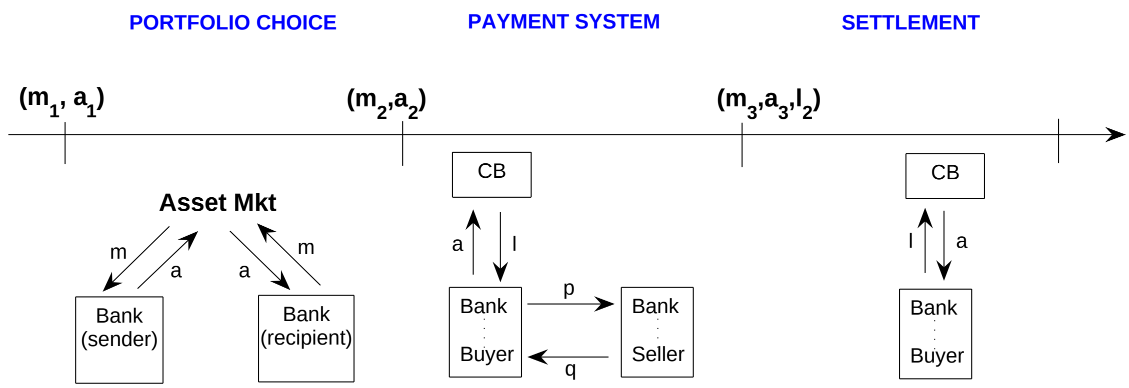
2 Overview of the Model

Before we formally describe the model, let us first briefly discuss the main features and the key findings. This is an infinite horizon model. Alternating over time are three types of markets: a centralized asset market (denoted AM), a decentralized goods market (denoted DM), and a centralized goods market (denoted CM). Our model bases on the alternating market formulation of Lagos and Wright (2005), and Berentsen and Monnet (2008), because it allows us to study frictions in the inter-bank market but still have frictionless trade in asset and other markets.

There is a continuum of infinitely lived anonymous agents. As in Berentsen and Monnet (2008), one can interpret each of these agents as a consolidated unit consisting of a bank and their clients.[1] In each period, agents participate in these three consecutive markets. In the first subperiod, agents make portfolio choice between liquid and illiquid assets in the AM. In the second sub-period, agents trade goods against the liquid assets in a decentralized fashion in the DM. We interpret it as banks sending payments to each other in the payment system to settle the goods transactions among their underlying clients. The central bank provides intra-day collateralized loans to agents subject to a haircut. In the third sub-period, agents enters the CM to trade a numeraire good and to settle their intra-day loans with the central bank.

To introduce the feature of portfolio choice, we assume that there are two assets: liquid and illiquid. The liquid asset is the only asset that is acceptable as a means of payment in the DM. It is denoted by m_t, and can be interpreted as fiat money or bank reserves. The supply of the stock of this asset is controlled by the central bank. The illiquid asset is

[1] We think that modeling the bank-client relationship explicitly is interesting, but may not be of first order importance for the main question of the paper. We will leave this extension for future research.



denoted by a_t. It is illiquid because it cannot be used as a means of payment in the DM. One can interpret it as claims to investment projects held by the agents. For simplicity, we assume that each agent is endowed with A one-period projects at the beginning of a period. Each unit yields certain amounts of CM numeraire goods at the end of a period (in the CM). To introduce the feature of asset price uncertainty, we assume that the return of the illiquid asset is a random i.i.d. (over time and across owners) variable. The price of these projects are denoted by ψ.

At the beginning of each period, each agent receives a noisy signal which suggests whether an agent is likely to be a payment sender (buyer) in the DM (i.e. high liquidity need), or likely to be a payment recipient (seller) in the DM (i.e. low liquidity need). Given the signal, agents trade in the AM and make portfolio choice of liquid asset m and illiquid asset a. Typically, an agent expecting high liquidity need will choose to hold more liquid asset, and one expecting low liquidity need will choose more illiquid asset.

To introduce idiosyncratic liquidity shocks, we assume that the signal will turn out to be incorrect with a positive probability. In particular, after the portfolio choice is made, an agent enters the DM and observes the realization of his/her trading status: buyer (i.e. payment sender) or a seller (i.e. payment recipient). Since trading in the DM is subject to liquidity constraint (only m is acceptable as means of payment), some agents will end up holding too much illiquid asset when they want to purchase goods. Their liquidity constraints can be relaxed by borrowing from the central bank's intra-day lending facility. Before trade, agents have access to this facility and borrow a nominal loan l by posting illiquid asset a, subject to a haircut h, implying the following borrowing constraint

$$l \leq a\psi_2(1-h),$$

where ψ_2 is the price of the asset when the loan is lent out in the second sub-period. This loan has to be settled in the CM in the third sub-period.[2] To introduce the role of strategic default, we assume that at the beginning of the CM, the values of all projects become public information, and after that borrowers decide whether to settle the loan (and get back the asset) or to default (and lose the asset). In the absence of additional punishment device, a borrower who has pledged asset a and borrowed l will default if[3]

$$l \geq a\psi_3,$$

where ψ_3 is the price of the asset when the loan is repaid in the third sub-period. So borrowers choose to default whenever the realization of the asset value is low.

We use this model to study the equilibrium effects of the haircut on the default decision, consumption, asset prices, portfolio choice and welfare. We found two key elements in determining the optimal level of the haircut.

First, the model implies that the lending facility is indeed providing a bundle of two insurances: an insurance against the liquidity risk and an insurance against the downside risk of the illiquid asset. On the one hand, lowering the haircut relaxes the liquidity constraint of the illiquid agents (which can be welfare improving). On the other hand, it also provides the borrowers an option to shift the investment loss to the central bank when the value of the asset turns out to be low (which is not welfare improving).[4] As a result, decreasing the

[2]Because intra-day loans are interest free in the LVTS, we also make this assumption here.

[3]In general, one can assume that default also involves a cost of R (e.g. punishment, reputation cost). As a result, an agent will default only if $l \geq a\psi_3 + R$. When R is a finite number, agents may still strategically default. When $R = +\infty$, agents have perfect commitment. When R is drawn randomly from the set $\{-\infty, +\infty\}$, then it is exogenous default. Futhermore, agents are assumed to be anonymous, so the central bank or other agents cannot induce repayment by future punishment (e.g. forever autarky). One may relax this assumption and endogenize the value of R. While the current setup is probably unrealistic, we try to study this extreme case as a benchmark, and leave other extensions for future research.

[4]Agents in our model have quasi-linear preference. The utility is concave in the consumption (the purcase

haircut will make illiquid asset more attractive, and may distort agents' portfolio choice: inducing an agent with high liquidity need to hold a portfolio which is illiquid.

Second, lowering the haircut will increase the exposure of the central bank. When taxation is not a an instrument available to the central bank, liquidity loaned out for payment may not be fully re-absorbed if the borrowers default. This will increase potential inflation, and the equilibrium opportunity costs of holding liquid assets, tightening the liquidity constraint of liquid asset holders.

3 Environment

(a) Sequence of Markets

Time is discrete and denoted $t = 0, 1, 2, ...$ In this economy, there is a measure one continuum of infinitely lived agents. Each period is divided into three consecutive sub-periods. In sub-period 1, there is an asset market (denoted by AM) for trading asset. In sub-period 2, there is a (decentralized) goods market (denoted by DM) for trading goods. In sub-period 3, there is an (centralized) goods market (denoted by CM) for settlement.

Markets, Value Functions and Prices in Sub-periods

Market	Value Function	Nominal Price of Asset	Real Price of Money
Subperiod 1: AM	$Z(m_1)$	ψ_1	N.A.
Subperiod 2: DM	$V^{DM}(m_2, a_2)$	N.A.	N.A.
AM	$V^{AM}(m_2, a_2)$	ψ_2	N.A.
Subperiod 3: CM	$W(y_3)$	ψ_3	ϕ_3

AM: Walrasian market for trading money and asset
DM: Bilateral bargaining for trading money and DM good
CM: Walrasian market for trading money and CM good

We are going to consider a stationary environment. The per-period utility of an agent is given by

$$u(q_2^b) - q_2^s - H_3,$$

of which is subject to a liquidity constraint), but is linear in the return of asset (which has random return). As a result, the insurance against the liquidity risk is welfare improving because of the concavity, but the insurance against the downside risk of the asset is not welfare improving because of the linearity.

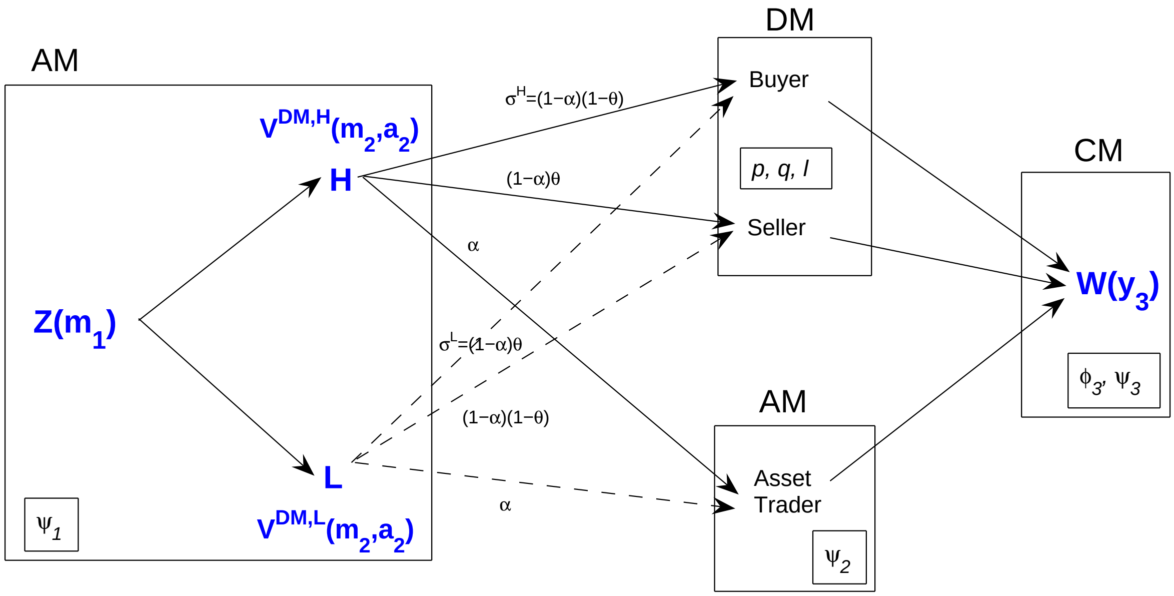


Figure 1: Timeline

where $q_2^b \in \mathbb{R}_+$ denotes the consumption of the DM goods when the agent is a buyer, and and $q_2^s \in \mathbb{R}_+$ denotes the production of the DM goods when the agent is a seller in the second sub-period. $u : \mathbb{R}_+ \to \mathbb{R}$ denotes the utility of consuming q units of the DM goods. $H_3 \in \mathbb{R}$ denotes the production (net of consumption) of the CM goods. We assume that $u(.)$ is twice continuously differentiable, strictly increasing, strictly concave, satisfies $u(0) = 0$, $u'(0) = \infty$, $u'(\infty) = 0$, $u'(q^*) = 1$ for some $q^* > 0$.

(b) Money and Asset

In this economy, there are two perfectly divisible, and costlessly storable objects which cannot be produced or consumed by any private individual: *fiat money* and *asset*. Money pays no dividends. Government injects money by lump sum transfers at a constant (gross) rate γ in the CM. At the beginning of each period t, each agent is initially endowed with A units of asset. Each unit of asset yields real dividend δ_t (in terms of CM goods) at the end of the period t CM. For simplicity, we will focus on one-period asset which is storable only within a period, but not across periods.

δ is a random i.i.d. (owner specific) variable, drawn from a *uniform* distribution over the

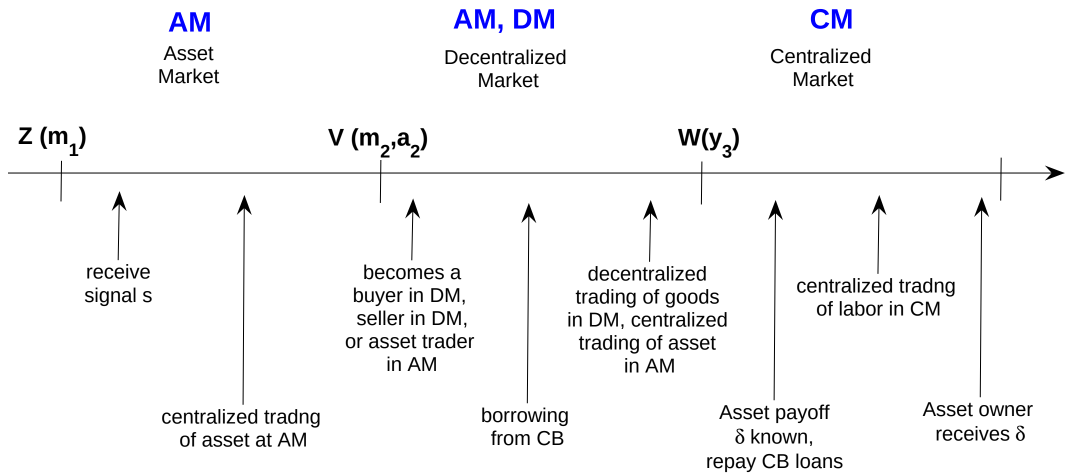


Figure 2: Sequence of Events

support $[\bar{\delta}(1-\varepsilon), \bar{\delta}(1+\varepsilon)]$, and with mean $\bar{\delta} < 1$. ψ_1 is the nominal price of the asset in the subperiod 1 AM. ψ_2 is the nominal value of the dividend of the asset in the subperiod 2 AM. And ψ_3 is the price in the subperiod 3 CM after the realization of δ (before the dividend is paid).

4 Solving the Model

We are going from solve the model in a downward fashion: first solving for the CM problem in sub-period 3, then the DM and AM problems in sub-period 2, and finally the AM problem in sub-period 1.

(a) Subperiod 3: Centralized market

We now start to discuss the three subperiods in a backward fashion.

In the CM, agents choose money holding (m_{+1}) for the following AM. The price of money in terms of CM goods is ϕ_3. We use y_3 to denote the real value of wealth an agent brings to the CM. Agent's optimization problem is to choose production H_3, and money holding m_{+1} to maximize payoff:

$$W(y) = \max_{H_3, m_{+1}} -H_3 + \beta Z_{+1}(m_{+1})$$

subject to

$$-H_3 = y_3 - \phi_3 m_{+1} + \phi_3(\gamma - 1)M.$$

Therefore,

$$W(y_3) = \max_{m_{+1}} y_3 - \phi_3 m_{+1} + \phi_3(\gamma - 1)M + \beta Z_{+1}(m_{+1})$$

F.O.C.s:

$$m_{+1} \quad : \quad \phi_3 \geq \beta \frac{\partial}{\partial m_{+1}} Z_{+1}(m_{+1}), = \text{if} \;\; m_{+1} > 0$$

Note that the choice of m_{+1} is independent of y_3. Assuming Z is strictly concave, we have $m_{+1} = M_{+1} = \gamma M$ for all agents, so that the distribution of money holding at the beginning of each period is degenerate. The envelope condition is given by

$$W'(y_3) = 1.$$

So

$$W(y_3) = W(0) + y_3$$

In principle, agents can also trade their assets in the CM. No arbitrage condition implies that an asset which is going to deliver δ units of goods at the end of the period is traded at a nominal price $\psi_3(\delta) = \delta/\phi_3$.

(b) Subperiod 2: Decentralized market

In subperiod 2, agents start with money holding m_2 and asset holding a_2. There is a shock that determines an agent's trading status. With a probability α, an agent enters the AM as an asset trader. With a probability $1 - \alpha$, an agent enters the DM as a goods trader.[5]

In the DM, an agent is either a buyer or a seller. In a bilateral meeting, the buyer makes a take-it-or-leave-it offer (q, p) to the seller, where q denotes the quantity of goods and p denotes the quantity of money to be traded. Before trade, buyers (but not sellers or asset traders) have

[5]We only need an asset market in the second sub-period to pin down the price of the asset. For this purpose, α has to be non-zero, but can be arbitrarily small.

access to central bank standing facilities. The loan is repaid in the next CM. The intra-day interest rate is 0. Therefore, a buyer can pay the price p by using his initial money holding (m_2) and/or by borrowing a nominal loan l_2 from the central bank standing facilities by posting the asset as collateral.

The borrowing constraint in nominal terms is set by the central bank:

$$l_2 \leq \psi_2 a_2 (1-h),$$

where $h \in [0, \varepsilon]$ is the haircut. Here, l_2 is the nominal repayment in the CM, $\psi_2 a_2$ is the nominal value of the asset in sub-period 2. The central bank sets a haircut h to make sure that the value of the collateralized asset in the CM will be sufficiently high to induce repayment (at least in certain realization of δ).

Before we consider the DM problem, let's first determine the continuation value in the next CM. At the beginning of the next centralized market, agents observe δ and $\psi_3(\delta) = \delta/\phi_3$ and choose whether to pay back l_2 or to give up the collateral and default. Note that the real wealth at the beginning of the following centralized market is

$$y_3 = \phi_3[m_2 - (p - l_2)] + \phi_3\psi_3[a_2 - \frac{l_2}{\psi_2(1-h)}] + \max\{\phi_3\psi_3\frac{l_2}{\psi_2(1-h)} - \phi_3 l_2, 0\}$$

Here, the real wealth is equal to the real value of the unspent money holding ($\phi_3(m_2 - (p - l_2))$), plus real value of the unpledged asset ($\phi_3\psi_3[a_2 - \frac{l_2}{\psi_2(1-h)}]$), plus the potential gain from repaying the loan ($\phi_3\psi_3\frac{l_2}{\psi_2(1-h)} - \phi_3 l_2$). Note that the agent always has an option to default, in particular it happens when the asset value drops too much (i.e. $\frac{\psi_3}{\psi_2}$ too low) relative to the haircut (i.e. $\frac{1}{1-h}$ too low). Simplifying the above expression, we get

$$y_3 = \phi_3 m_2 - \phi_3 p + \phi_3\psi_3 a_2 + \max\{0, \phi_3 l_2 - \phi_3\psi_3\frac{l_2}{\psi_2(1-h)}\}$$

If the central bank wants to ensure repayment in *any* circumstances, the following inequality has to be satisfied for any δ:

$$\begin{aligned}\phi_3 l_2 - \phi_3\psi_3(\delta)\frac{l_2}{\psi_2(1-h)} &\leq 0 \\ \text{or } \frac{\psi_2 - \psi_3(\delta)}{\psi_2} &\leq h\end{aligned}$$

Therefore, the no-default constraint is particularly binding when ψ_3 (i.e. δ) is low. When $\delta = \bar{\delta}$ in all realization (i.e. $\varepsilon = 0$), h can be set to zero (no haircut). When $\psi_3 = 0$ in some realization (i.e. $\varepsilon = 1$), h has to be one (i.e. the asset cannot be pledged as a collateral) to satisfy the no-default constraint. In general, if the central bank set an haircut such that

$$h < 1 - \frac{\psi_3(\delta)}{\psi_2},$$

then there will be default when δ is sufficiently low.

As a result,

$$y_3 = \begin{cases} \phi_3 m_2 - \phi_3 p + \phi_3\psi_3 a_2 + l_2 \max\{\phi_3 - \frac{\phi_3\psi_3}{\psi_2(1-h)}, 0\} & \text{, for a buyer} \\ \phi_3 m_2 + \phi_3 p + \phi_3\psi_3 a_2 & \text{, for a seller} \end{cases}$$

Therefore, the payoff of a buyer in the DM is

$$\begin{aligned} & u(q) + EW(y) \\ = \; & u(q) + EW(0) + E(y) \\ = \; & u(q) + EW(0) + \phi_3 m_2 - \phi_3 p + \phi_3 a_2 E(\psi_3) + l_2 E \max\{\phi_3 - \frac{\phi_3\psi_3}{\psi_2(1-h)}, 0\} \\ = \; & u(q) + \text{constant} + \phi_3 m_2 - \phi_3 p + \phi_3 a_2 E(\psi_3) + \phi_3 l_2 S(h), \end{aligned}$$

where $S(h) = \frac{(\varepsilon-h)^2}{4\varepsilon(1-h)}$ is derived in appendix A. Note that, $S \geq 0$ and S is positive whenever $h < \varepsilon$ (i.e. partial haircut). Now, we look at the bargaining problem in the decentralized market when the standing facility is available:

$$\max_{q,p} u(q) + (\phi_3 m_2 - \phi_3 p + \phi_3 a_2 E(\psi_3) + \phi_3 l_2 S(h))$$

subject to

$$\begin{aligned} \text{Liquidity constraint} &: \quad m_2 + l_2 \geq p \\ \text{Borrowing constraint} &: \quad l_2 \leq \psi_2 a_2 (1-h) \\ \text{Seller's participation constraint} &: \quad \phi_3 p = q \end{aligned}$$

This is equivalent to solving

$$\max_{q,l_2} u(q) - q + \phi_3 l_2 S(h)$$

subject to

$$\begin{aligned} m_2 + l_2 &\geq q/\phi_3 \\ \psi_2 a_2 (1-h) &\geq l_2 \end{aligned}$$

Using η_m and η_a to denote the multipliers of the two constraints, then the FOCs are given by

$$\begin{aligned} q &: \quad \phi_3 (u'(q) - 1) = \eta_m \\ l_2 &: \quad \phi_3 S(h) + \eta_m = \eta_a \end{aligned}$$

From now on, we will focus on monetary equilibria with $\phi_3 > 0$. The first condition implies that whenever $u'(q) > 1$, the liquidity constraint is binding. The second condition implies that whenever $S(h) > 0$ or $u'(q) > 1$, the borrowing constraint is binding.

The previous finding suggests that, if $h < \varepsilon$, then $l_2 = \psi_2 a_2 (1-h)$. That is, whenever the haircut is partial, buyers will borrow up to the borrowing limit. And the bargaining solution implies

$$q(m_2, a_2) = \begin{cases} q^*, & \text{if } \phi_3 m_2 + \phi_3 \psi_2 a_2 (1-h) \geq q^* \\ \phi_3 m_2 + \phi_3 \psi_2 a_2 (1-h), & \text{if } \phi_3 m_2 + \phi_3 \psi_2 a_2 (1-h) < q^* \end{cases}$$

where q^* satisfies $u'(q) = 1$. Denote the solution by $q(m_2, a_2)$.

(c) Subperiod 2: Asset market

In sub-period 2, with a probability α, an agent enters the AM as an asset trader. Their optimization problem is given by

$$V^{AM}(m_2, a_2) = \max_{m_3, a_3} W(y_3) = \phi_3 m_3 + \phi_3 E(\psi_3) a_3 + W(0)$$

subject to

$$(\lambda_2) m_2 + \psi_2 a_2 = m_3 + \psi_2 a_3$$

F.O.C.s:

$$\begin{aligned} m_3 &: \lambda_2 \geq 1, = \text{if } \ m_3 > 0 \\ a_3 &: \lambda_2 \psi_2 \geq E(\psi_3), = \text{if } \ a_3 > 0 \end{aligned}$$

Market clearing conditions imply $\psi_2 = E(\psi_3)$. Note that the choices (m_3, a_3) are independent of (m_2, a_2).

$$\begin{aligned} V_m^{AM}(m_2, a_2) &= \phi_3 \\ V_a^{AM}(m_2, a_2) &= \phi_3 E(\psi_3) \end{aligned}$$

Therefore,

$$\begin{aligned} V^{AM}(m_2, a_2) &= \phi_3 m_3 + \phi_3 E(\psi_3) a_3 + W(0) \\ &= \phi_3 m_3 + \phi_3 \psi_2 a_3 + W(0) \\ &= \phi_3 m_2 + \phi_3 E(\psi_3) a_2 + W(0) \end{aligned}$$

Trading in AM does not affect the payoff of agents.

(d) Subperiod 2 value function

$$\begin{aligned} V^j(m_2, a_2) &= (1-\alpha) V^{DM,j}(m_2, a_2) + \alpha V^{AM}(m_2, a_2) \\ &= \phi_3 m_2 + \phi_3 E(\psi_3) a_2 + W(0) + \sigma^j [u(q(m_2, a_2)) - q(m_2, a_2)] \\ &\quad + \sigma^j [\phi_3 l_2 S(h)], \text{ for } j = H, L, \end{aligned}$$

Again, we have shown that $l_2 = \psi_2 a_2(1-h)$ if $h < \varepsilon$ or if $\phi_3 m_2 + \phi_3\psi_2 a_2(1-h) < q^*$. So the envelope conditions are,

$$V_m^j(m_2, a_2) \quad = \quad \phi_3 + \sigma^j[u'(q(m_2,a_2)) - 1]\phi_3 \mathbb{1}\{\phi_3 m_2 + \phi_3\psi_2 a_2(1-h) < q^*\}$$

$$\begin{aligned} V_a^j(m_2, a_2) \quad &= \quad \phi_3 E(\psi_3) + \sigma^j[u'(q(m_2,a_2)) - 1]\phi_3\psi_2(1-h)\mathbb{1}\{\phi_3 m_2 + \phi_3\psi_2 a_2(1-h) < q^*\} \\ &\quad + \sigma^j\phi_3\psi_2(1-h)S(h)\mathbb{1}\{\phi_3 m_2 + \phi_3\psi_2 a_2(1-h) < q^* \text{ or } h < \varepsilon\}, \end{aligned}$$

We will focus on equilibrium in which the liquidity constraint is binding. Therefore:

$$\begin{aligned} V_m^j(m_2, a_2) \quad &= \quad \phi_3 + \sigma^j\Delta^j\phi_3 \\ V_a^j(m_2, a_2) \quad &= \quad \phi_3 E(\psi_3) + \sigma^j[\Delta^j + S(h)]\phi_3\psi_2(1-h), \end{aligned}$$

where $\Delta^j = u'(q(m_2,a_2)) - 1$.

(e) Subperiod 1: Asset market

At the beginning of a period, each agent receives a signal $s \in \{H, L\}$. A signal H denotes the case in which the agent will likely become a buyer in the DM (high liquidity need). A signal L denotes the case in which the agent will likely become a seller in the DM (low liquidity need). The signal will turn out to be incorrect with a probability $\theta < \frac{1}{2}$. Therefore, an agent with a high signal will be a buyer with a probability $\sigma^H = (1-\alpha)(1-\theta)$, and an agent with a low signal will be a buyer with a probability $\sigma^L = (1-\alpha)\theta$. And an agent will attend the asset market with a probability α. After receiving the signal s, an agent solves the following portfolio choice problem:

$$\max_{m_2, a_2} V^j(m_2^j, a_2^j)$$

subject to

$$
\begin{aligned}
m_1 + \psi_1 A &\geq m_2^j + \psi_1 a_2^j (\text{with multiplier } \lambda^j) \\
m_2^j &\geq 0, \\
a_2^j &\geq 0.
\end{aligned}
$$

F.O.C.s:

$$
\begin{aligned}
m_2^j &: \lambda^j \geq V_m^j(m_2^j, a_2^j), \ (= \text{if } m_2^j > 0) \\
a_2^j &: \lambda^j \psi_1 \geq V_a^j(m_2^j, a_2^j), \ (= \text{if } a_2^j > 0)
\end{aligned}
$$

And the envelope conditions of the second sub-period are

$$
\begin{aligned}
V_m^j(m_2, a_2) &= \phi_3 + \sigma^j \Delta^j \phi_3 \\
V_a^j(m_2, a_2) &= \phi_3 E(\psi_3) + \sigma^j [\Delta^j + S(h)] \phi_3 E(\psi_3)(1-h)
\end{aligned}
$$

So, depending on whether the non-negativity constraints are binding or not, there are three different cases:

CASE ONE:

$$
\left.
\begin{array}{c}
V_a^j / \psi_1 (= \lambda^j) \geq V_m^j \\
\Rightarrow E(\psi_3) + \sigma^j [\Delta^j + S(h)] E(\psi_3)(1-h) \geq \psi_1 + \sigma^j \Delta^j \psi_1 \\
\Rightarrow E(\psi_3)[1 + S(h)(1-h)] - \psi_1 \geq \sigma^j \Delta^j [\psi_1 - E(\psi_3)(1-h)] \\
\Rightarrow Q(\sigma^j) = \frac{E(\psi_3)[1+\sigma^j S(h)(1-h)] - \psi_1}{\psi_1 - E(\psi_3)(1-h)} \geq \sigma^j \Delta^j
\end{array}
\right. \quad \text{if } a_2^j > 0, m_2^j = 0
$$

CASE TWO:

$$
\begin{aligned}
V_m^j(=\lambda^j) &\geq V_a^j/\psi_1 \\
\Rightarrow \psi_1[1+\sigma^j\Delta^j] \geq E(\psi_3)[1+\sigma^j[\Delta^j+S(h)](1-h)] & \quad \text{if } a_2^j=0, m_2^j>0 \\
\Rightarrow \sigma^j\Delta^j &\geq Q(\sigma^j)
\end{aligned}
$$

CASE THREE:

$$
\begin{aligned}
V_a^j/\psi_1 &= V_m^j(=\lambda^j) \\
\Rightarrow \psi_1[1+\sigma^j\Delta^j] = E(\psi_3)[1+\sigma^j[\Delta^j+S(h)](1-h)] & \quad \text{if } a_2^j>0, m_2^j>0 \\
\Rightarrow \sigma^j\Delta^j &= Q(\sigma^j)
\end{aligned}
$$

Finally, the envelope condition in the first sub-period is given by:

$$
\begin{aligned}
Z_m(m_1,a_1) &= \frac{1}{2}Z_m^H(m_1,a_1)+\frac{1}{2}Z_m^L(m_1,a_1)=\frac{1}{2}(\lambda^H+\lambda^L) \\
Z_a(m_1,a_1) &= \frac{1}{2}Z_a^H(m_1,a_1)+\frac{1}{2}Z_a^L(m_1,a_1)=\frac{1}{2}\psi_1(\lambda^H+\lambda^L)
\end{aligned}
$$

And the market clearing conditions are:

$$
\begin{aligned}
M &= \frac{1}{2}m_2^H+\frac{1}{2}m_2^L \\
A &= \frac{1}{2}a_2^H+\frac{1}{2}a_2^L
\end{aligned}
$$

5 Characterization of Equilibrium

In this section, we will characterize the steady state equilibrium given the policy set by the government: M, A, γ. For simplicity, we will first look at equilibrium in which the liquidity constraints are binding for both types. We will focus on the steady state equilibrium in which nominal prices are growing at the rate of money growth, and real quantities are constant over time: $\frac{\phi}{\phi_{+1}} = \frac{\psi_{+1}}{\psi} = \gamma$, and $q = q_{+1}$.

In particular, a steady state equilibrium can be defined as (m_2^H,m_2^L,a_2^H,a_2^L,q^H, q^L,ϕ_3, ψ_1,ψ_3, λ^H, λ^L) satisfying the following set of conditions[6] We will use superscripts a and m to denote the two types who hold A and M (in case one holds both assets, w.l.o.g., let's use m to denote H and a to denote L in this case[7])

Equilibrium conditions:

$$
\begin{aligned}
\phi_3 &= \beta Z_{+1,m} = \beta \frac{1}{2}(\lambda^a_{+1} + \lambda^m_{+1}) && (1)\\
q^a &= \phi_3 m_2^a + \phi_3 E(\psi_3) a_2^a (1-h) && (2)\\
q^m &= \phi_3 m_2^m + \phi_3 E(\psi_3) a_2^m (1-h) && (3)\\
m_2^a + m_2^m &= 2M && (4)\\
a_2^a + a_2^m &= 2A && (5)\\
\lambda^m &= V_m^m = \phi_3(1+\sigma^m \Delta^m) && (6)\\
\lambda^a &= V_a^a/\psi_1 = \phi_3 E(\psi_3)[1+\sigma^a(\Delta^a + S(h))(1-h)]/\psi_1 && (7)\\
m_2^m + \psi_1 a_2^m &= M + \psi_1 A && (8)\\
\psi_3(\delta)\phi_3 &= \delta && (9)\\
Q(\sigma^a) &\geq \sigma^a \Delta^a && (10)\\
\sigma^m \Delta^m &\geq Q(\sigma^m) && (11)
\end{aligned}
$$

Here, (1) is the condition for the optimal money demand in the CM. (2) and (3) are the binding liquidity constraints in the DM. (4) and (5) are the market clearing conditions in the first subperiod AM. (6),(7), (10) and (11) are conditions for the optimal portfolio choice in the first sub-period.

[6]Note that we have shown $m_1^H = m_1^L = M$, $\psi_2 = E(\psi_3)$ in equilibrium.

[7]And both (10) and (11) have to be satisfied for a type holding both money and asset.

(8) is the budget constraint in the first sub-period. (9) is the market price of an asset that delivers δ.

Defining i as the (net) nominal interest rate, then the Fisher's equation and (1) imply

$$1 + i = \frac{\gamma}{\beta} = \frac{\phi_3}{\beta \phi_{3,+1}} = \frac{1}{2}(\frac{\lambda_{+1}^m + \lambda_{+1}^a}{\phi_{3,+1}})$$

(6) and (7) then imply

$$1 + i = \frac{1}{2}\left[\{1 + \sigma^m[u'(q^m) - 1]\} + \frac{E(\psi_{3,+1})}{\psi_{1,+1}}\{1 + \sigma^a[u'(q^a) - 1 + S(h)](1 - h)\}\right] \quad (12)$$

(8) implies

$$\psi_1 = \frac{(m_2^m - M)}{(A - a_2^m)}$$

Combining (2)-(5), (9) and (12) gives one equation in terms of (ϕ_3, m_2^a, a_2^a):

$$1 + i = \frac{1}{2}\left[\begin{array}{c} \{1 + \sigma^m[u'(\phi_3(2M - m_2^a) + \bar{\delta}(2A - a_2^a)(1 - h)) - 1]\} \\ + \frac{\bar{\delta}}{\phi_3 \psi_1}\{1 + \sigma^a[u'(\phi_3 m_2^a + \phi_3 E(\psi_3) a_2^a (1 - h)) - 1 + S(h)](1 - h)\} \end{array}\right], \quad (13)$$

where $\psi_1 = \frac{(m_2^m - M)}{(A - a_2^m)}$. Therefore, an equilibrium is given by (ϕ_3, m_2^a, a_2^a) that satisfies equation (13) and conditions (10)-(11).

6 Policy Constraint

In the previous section, we characterize the set of equilibrium given any arbitrary policy i (i.e. which is pinned down by the money growth rate γ) and h. However, not all (i, h) policy pairs are feasible for the central bank to pick. In particular, the choice of h will imply a minimum size of money injection, and thus a minimum level of interest rate i.

Note that whenever a buyer defaults its loan l_2, the new money temporarily lent out by the central bank in sub-period 2 will only be partially withdrawn by the central bank who sells the asset for $\psi_3(\delta) = \delta/\phi_3$ in sub-period 3.

For each unit of asset posted as collateral, the expected nominal size of default is (see Appendix A):

$$E \max\{\psi_2(1-h) - \psi_3, 0\} = \frac{\bar{\delta}}{4\phi_3\varepsilon}(\varepsilon - h)^2$$

Let $\bar{A}$ be the amount of asset posted as collateral, the (net) money growth due to unrepaid loans is

$$\gamma - 1 = \bar{A}\frac{\bar{\delta}}{4M\phi_3\varepsilon}(\varepsilon - h)^2. \tag{14}$$

7 Equilibrium (Special Case)

In this section, we will first consider one simple equilibrium in which the H-type brings only money and the L-type brings only asset to the second sub-period: $m_2^H = m_2^m = 2M, a_2^H = a_2^m = 0$. For simplicity, assume that $u(q) = \log(q)$.

Equation (13) then implies that (see Appendix B)

$$\phi_3 = \frac{1}{2M(1+2i+\sigma^H)}\left[(1-\alpha)+2\bar{\delta}A\left(1-\sigma^L[1-h-\frac{\bar{\delta}}{2}(\varepsilon-h)^2]\right)\right] \tag{15}$$

The equilibrium quantities are

$$q^H = 2M\phi_3$$
$$q^L = 2A\bar{\delta}(1-h)$$

The equilibrium prices are

$$\psi_1 = \frac{M}{A}$$
$$\psi_2 = E(\phi_3) = \frac{\bar{\delta}}{\phi_3}$$

The welfare measured by ex-ante expected utility is

$$W(i,h) = \sigma^H(\log(q^H) - q^H) + \sigma^L(\log(q^L) - q^L)$$

The policy constraint (14) is given by

$$i = \frac{\gamma}{\beta} \geq \frac{\sigma^L A\bar{\delta}}{4\beta M\phi_3\varepsilon}(\varepsilon-h)^2 + \frac{1}{\beta} - 1. \tag{16}$$

1. Effect of an increase in i

If (16) is not binding, an increase in the interest rate i lowers the equilibrium value of money (ϕ_3) (by (15)), and lowers the equilibrium consumption of the H-type (q^H), and reduces the average welfare.

2. Effect of an decrease in h

A cut in the haircut h relaxes the borrowing constraint of the L-type and thus increases the equilibrium consumption of the L-type (q^H). Given that $\bar{\delta} < 1$, a cut in hair cut will lead to a lower ϕ_3 (by (15))[8] and thus lower consumption of the H-type.

If (16) is initially binding, a cut in h will also tighten the policy constraint (by (16)), raising the lower-bound of the interest rate, which will further reduce the consumption of the H-type.

Here, we can see that lowering the haircut has different effects on agents with different portfolio choices. On the one hand, it can relax the liquidity constraint of illiquid asset holders. On the other hand, it will lower the value of liquid assets (e.g. money) by both reducing the returns to holding liquidity and increasing the cost of holding liquidity (by increasing i). As a result, it will tighten the liquidity constraint of liquid asset holders.

3. Effect of a drop in $\bar{\delta}$ or A

An drop in $\bar{\delta}$ or A will lower the consumption of the L-type, and it will also decrease the value of ϕ_3 and thus lower the consumption of the H-type. If (16) is initially binding, it will relax the policy constraint and allow for a higher h or a lower i.

4. Effect of increase in ε

An increase in ε will not affect the consumption of the L-type, but it will increase the value of ϕ_3 and thus increases the consumption of the H-type. If (16) is initially binding, it will tighten the policy constraint and require a higher h or a higher i to satisfy the policy constraint.

7.1 Numerical Examples

In this section, we will use a numerical example to illustrate how the model implications derived above. In particular, we will set the parameter values as follows: $M = 1$, $A = 7.5$, $\beta = 0.94$, $\bar{\delta} = 0.06$, $\varepsilon = 0.4$, $\alpha = 0.1$, $\theta = 0.02$.

[8]The marginal effect of h on ϕ_3 is given by sign$[\frac{\mathrm{d}}{\mathrm{d}h}\phi_3]$=$1 - \bar{\delta}(\varepsilon - h) > 0$.

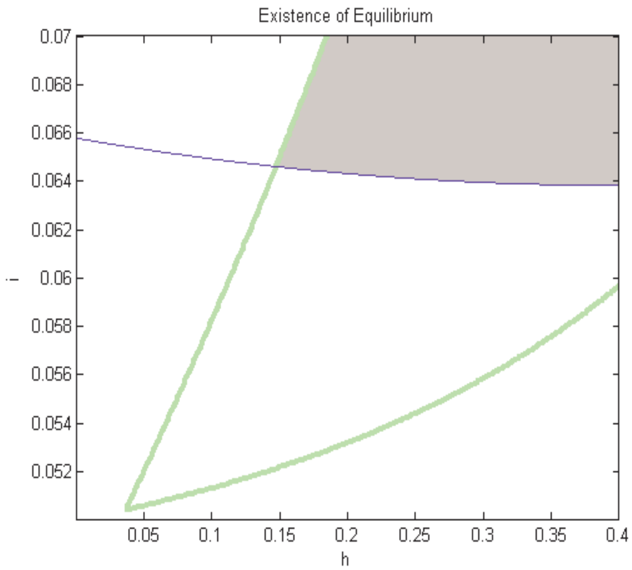


Figure 3: Existence of Equilibrium

Equilibrium

The previous analysis shows that a simple equilibrium exists when equations (10), (11), (15) and condition (16) are all satisfied. Figure 3 shows the existence of equilibrium over the (h, i) plain. In particular, equations (10), (11), (15) are satisfied inside the area bounded by the green line. Condition (16) is satisfied for any (h, i) pairs lying above the blue curve. As shown above, the policy constraint is downward sloping. Therefore, inside the grey area, a simple equilibrium exists.

Choice of policy

The choice of (h, i) depends on the preference of the policy maker. The consumption of H-type is increasing in h and decreasing in i. The consumption of L-type is decreasing in h and is independent of i. In our example, the total output and the welfare are both decreasing in h and i. It turns out that, in order to maximize the consumption of the H-type, the policy maker should choose $h = \varepsilon = 0.4000$ and $i = 0.0638$. Alternatively, a policy maker who wants to maximize the consumption of the L-type, the total output or the welfare should set $h = 0.1480$ and $i = 0.0646$ (**choosing within the set of simple equilibrium**).

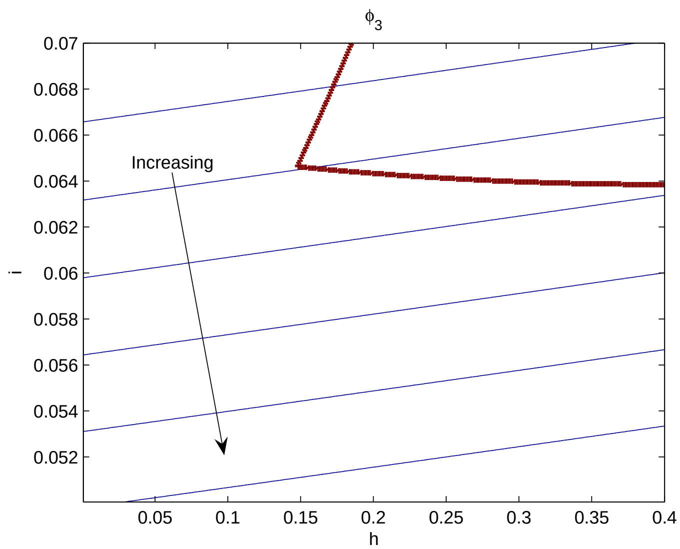


Figure 4: Real Price of Money

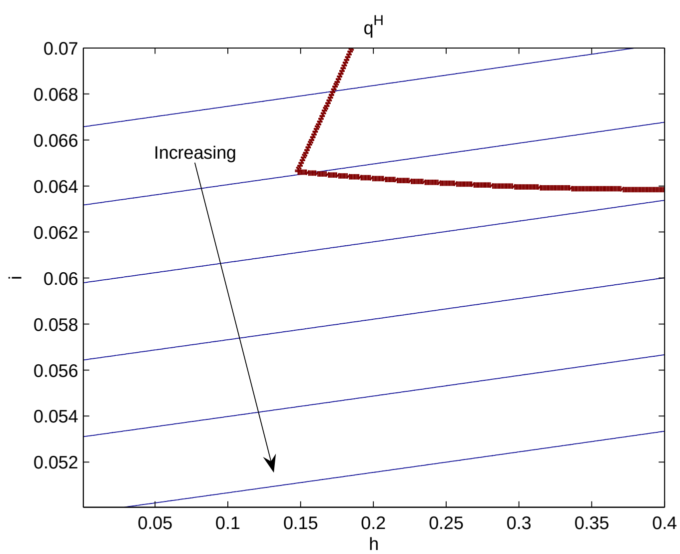


Figure 5: Consumption of H-type

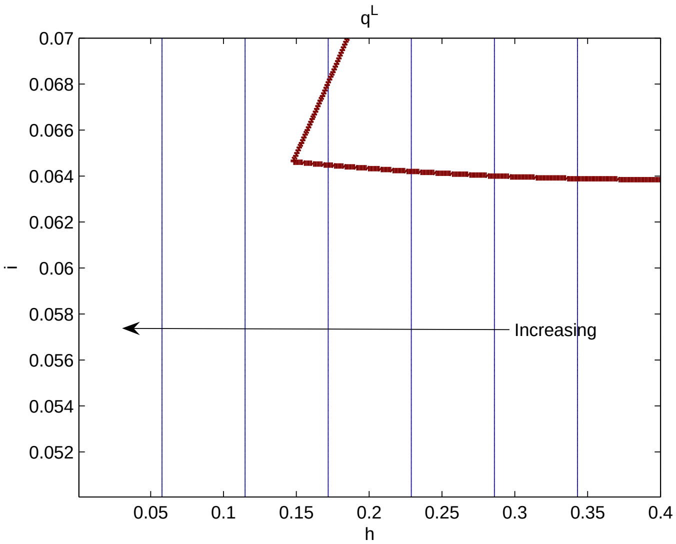


Figure 6: Consumption of L-type

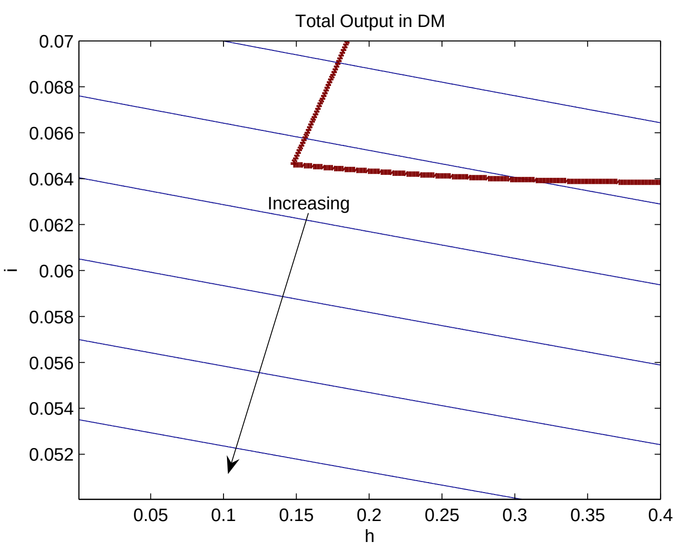


Figure 7: Total Consumption

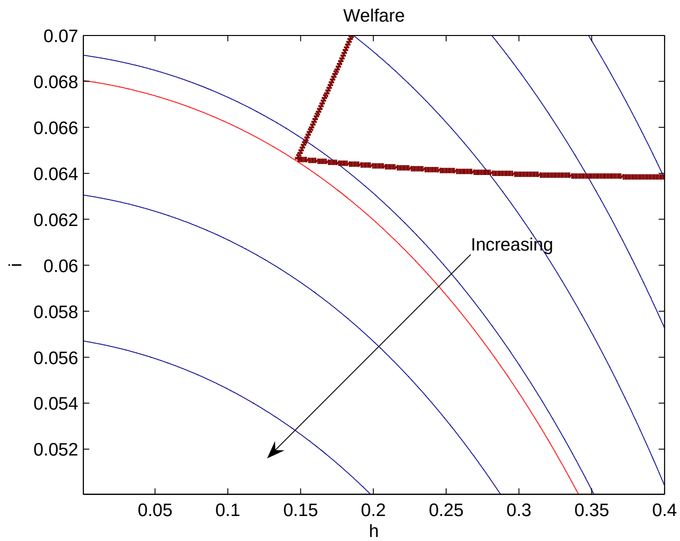


Figure 8: Welfare

7.2 Extensions

Non-exclusive Lending Facility

In the previous sections, we assume that the central bank is able to lend exclusively to buyers who are in need of liquidity. Now, suppose the central bank is unable to exclude sellers from borrowing from the lending facility, then the equilibrium value of money is modified to

$$\phi_3 = \frac{1}{2M(1+2i+\sigma^H)}\left[1-\alpha+2\bar{\delta}A(1-(1-h)\sigma^L)+\bar{\delta}^2A(1-\alpha)(\varepsilon-h)^2\right].$$

And the portfolio choice is also modified to

$$Q(\sigma^H)=Q(\sigma^L)=\frac{E(\psi_3)[1+(1-\alpha)S(h)(1-h)]-\psi_1}{\psi_1-E(\psi_3)(1-h)}.$$

The policy constraint becomes

$$i \geq \frac{(1-\alpha)A\bar{\delta}}{4\beta M\phi_3\varepsilon}(\varepsilon-h)^2+\frac{1}{\beta}-1.$$

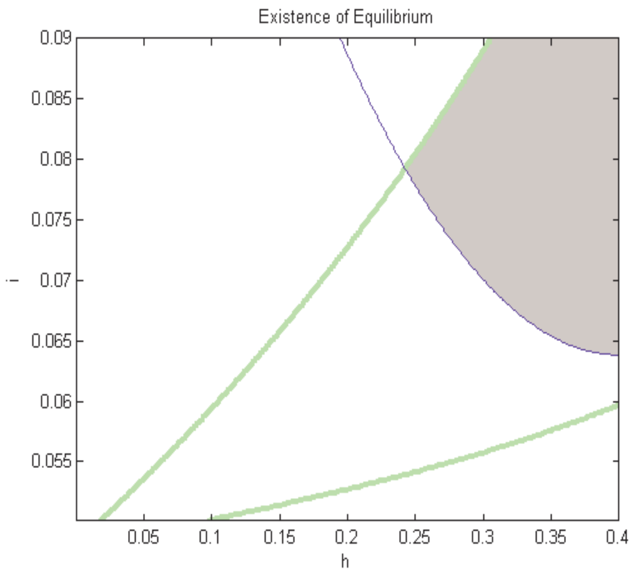


Figure 9: Non-exclusive Lending: Existence

Comparing this policy constraint with constraint (16) suggests that the policy constraint is tightened when the central bank cannot restrict lending to the buyers only: the i lower bound is higher for any given h, and the marginal effect of h on the i lower bound is higher. Therefore, in Figure 9, the feasible set of policy becomes smaller.

The welfare maximizing policy is given by $i = 0.0671$ and $h = 0.3264$. When the central bank cannot restrict lending, the cost of providing consumption insurance to the L-type by lowering the haircut h becomes more costly in terms of distorting the H-type's consumption. Under this trade-off, it is generally not optimal to set the highest or lowest possible haircut.

Temporary versus Permanent Change in Haircut

In the previous section, we consider optimal permanent changes in haircut. Here, we study how a one-time change in haircut can improve on the allocation temporarily.

Here, we consider the case in which the central bank can exclude borrowing from the sellers. Suppose the central bank is following the optimal policy (i.e. $h = 0.1480$, $i = 0.0646$) and is allowed to make one-time change in h in the current period (with the agents believing the central bank will bring the (i, h) back to the original levels before the change).

Since this is a one-time change in h, it will have no effect on future allocation. In particular, it

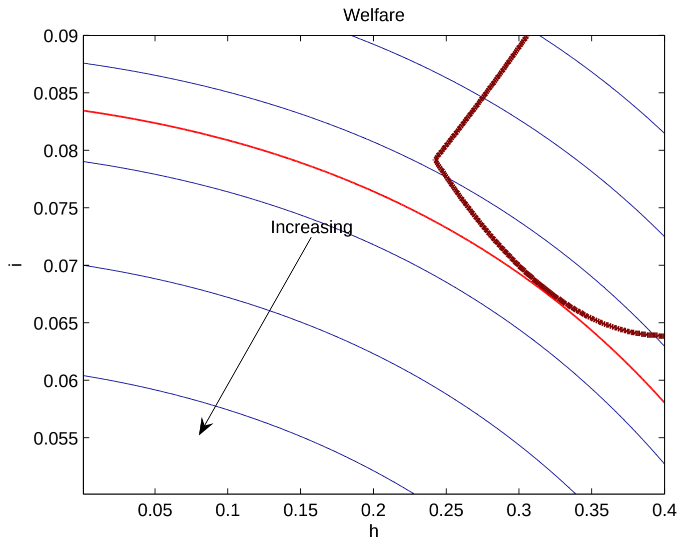


Figure 10: Non-exclusive Lending: Welfare

will not affect the policy constraint (16). The only effect is on the current stock of money supply and on the current price of money in the third sub-period. Denoting $\tilde{h}$ as the haircut in the current period, the current period equilibrium $\{\phi_3(\tilde{h}), \Delta M(\tilde{h}), q^H(\tilde{h}), q^L(\tilde{h})\}$ is then determined by

$$\begin{aligned}
\phi_3(\tilde{h}) &= \frac{(1-\alpha) + 2\bar{\delta}A\left(1 - \sigma^L[1 - h - \frac{\bar{\delta}}{2}(\varepsilon - h)^2]\right)}{2(M + \Delta M(\tilde{h}))/\gamma(1 + 2i + \sigma^H)} \\
\Delta M(\tilde{h}) &= \frac{\sigma^L A\bar{\delta}}{4\phi_3\varepsilon}(\varepsilon - \tilde{h})^2 \\
q^H(\tilde{h}) &= 2M\phi_3(\tilde{h}) \\
q^L(\tilde{h}) &= 2A\bar{\delta}(1 - \tilde{h}).
\end{aligned}$$

As shown in Figure (11), it is welfare maximizing to temporarily lower the hair cut from $h = 0.1480$ to $h = -0.0.0452$. Note that the optimal one-period deviation of the haircut is indeed negative to improve ex-post efficiency. This will temporarily increase the money stock (additionally by 0.16%), lower the price of money (by 0.15%), increase the consumption of the L-type (by 22.68%) and lower the consumption of the H-type (by 0.15%).

In the table, we also report the case of a one-time change in the haircut in the case of non-

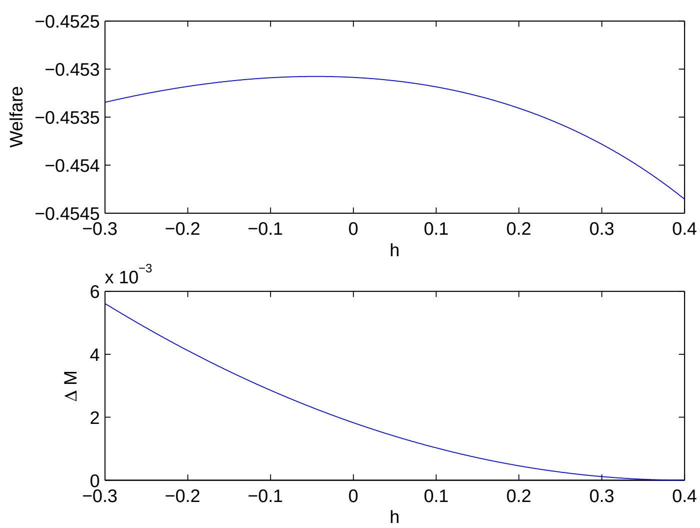


Figure 11: One-time Change in Haircut: Welfare and Change in Money Stock

exclusive lending facilities. As expected, in this case, it is not optimal to lower the haircut by too much.

	h	i	W	$\tilde{h}$	$\tilde{U}$
Exclusive Lending	0.1480	0.0646	-0.4533	-0.0452	-0.4531
Non-Exclusive Lending	0.3264	0.0671	-0.4540	0.3128	-0.4540

8 Equilibrium (General Case)

Figure 12 plots the equilibrium outcomes for different combinations of h and i. When the haircut h is high (the right portion of the graph), the H-type will bring only money to the second sub-period, and the L-type will bring only asset to the second sub-period. For lower haircut (the middle portion of the graph), the H-type is induced to hold both money and asset because of the higher value of the "default option" $S(h)$. For even lower haircut (the left portion of the graph), the H-type choose to hold only asset and the L-type choose to hold only money. Figure 13 plots the welfare over the (h, i) plain.

In figure 14, the blue downward sloping curve denotes the policy constraint. The grey curves denotes the indifference curves. The red dot indicates the welfare-maximizing policy which is given by $h = 0.16$ and $i = 0.065$. Starting from a high haircut, as the haircut goes down, the welfare is

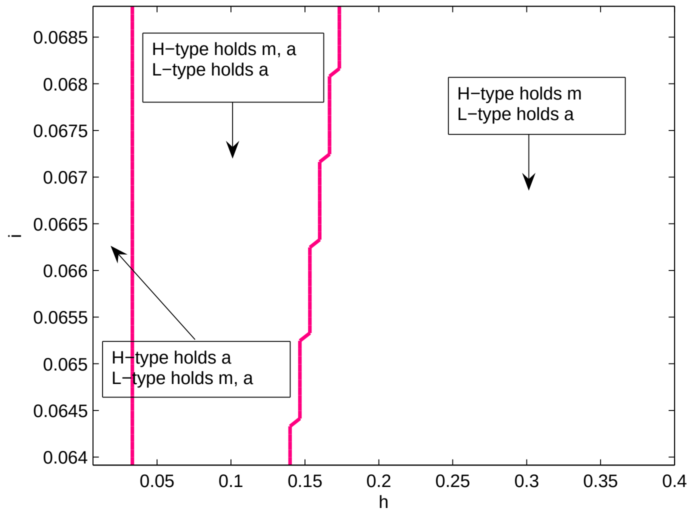


Figure 12: Distribution of Equilibria

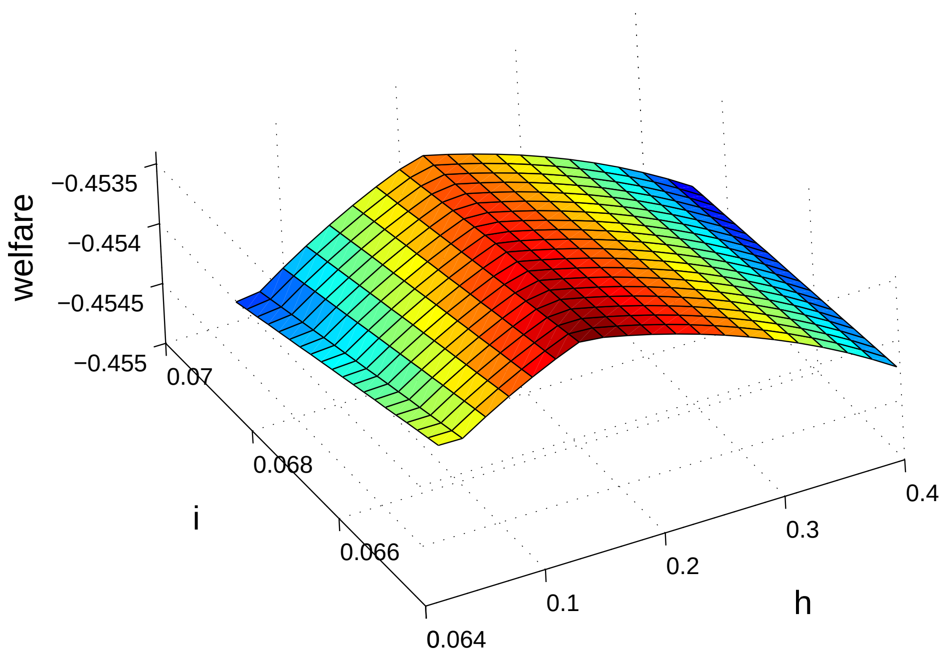


Figure 13: Welfare

improving because of the relaxation of the liquidity constraint. As the haircut goes down further, the welfare decreases for two reasons. First, the portfolio choice of agents is distorted. Second, the policy constraint starts to bind, and the nominal interest rate has to increase, lowering the welfare.

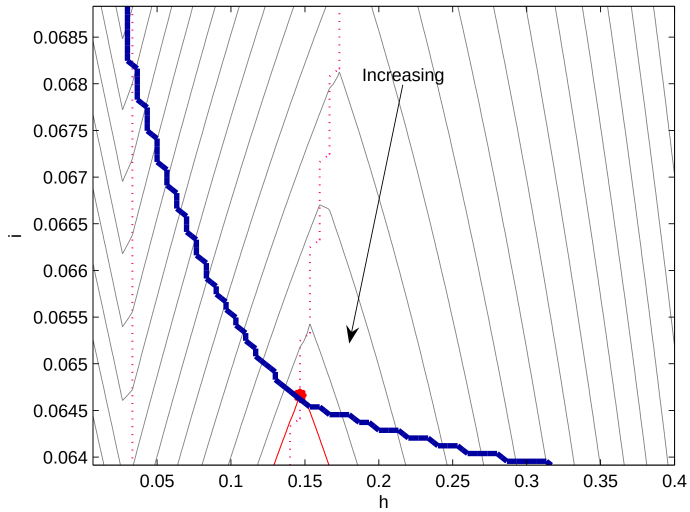


Figure 14: Welfare-maximizing Policy

9 Conclusion

We have constructed a simple analytically tractable model of a central bank liquidity facility. Due to the risk of default the central bank liquidity loans are collateralized. One of the policy instruments that is open to the central bank is the haircut (or discount) applied to the illiquid collateral pledged by borrowers.

Our analysis makes clear the twin types of insurance that a central bank liquidity offers. Namely, the insurance against a liquidity shock and also insurance against downside risk of the asset price. Therefore, the haircut on an asset must be positive to ensure that the borrower does not default on the loan.

In addition, we show that there is a trade-off between a lower haircut and lowering the interest cost of holding liquidity. As the haircut is lowered relative to the interest cost then the value of the liquid asset decreases along two channels. First, defaults on the collateralized loans cause injections of the liquid asset lowering its value. Second, lowering the haircut relative to the interest rate makes the illiquid asset less costly to hold (i.e. more liquid) thereby reducing the value of the liquid asset.

In our model the optimal choice of a haircut then depends on the sizes of the different groups, the tightness of agents' liquidity constraints, and the volatility of asset prices, and the exclusivity of the lending facility.

References

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Lagos, R., and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113, 463–484.

Appendix A

$$
\begin{aligned}
& l_2 E \max\{\phi_3 - \frac{\phi_3\psi_3}{\psi_2(1-h)}, 0\} \\
= \quad & \frac{\phi_3 l_2}{\psi_2(1-h)} E \max\{\psi_2(1-h) - \psi_3, 0\} \\
= \quad & \frac{l_2}{\psi_2(1-h)} E \max\{\bar{\delta}(1-h) - \delta, 0\} \\
= \quad & \frac{\phi_3 l_2}{\bar{\delta}(1-h)} E \max\{\bar{\delta}(1-h) - \delta, 0\} \\
= \quad & \frac{\phi_3 l_2}{2\varepsilon\bar{\delta}^2(1-h)} \int_{\bar{\delta}(1-\varepsilon)}^{\bar{\delta}(1-h)} [\bar{\delta}(1-h) - \delta] d\delta \\
= \quad & \frac{\phi_3 l_2}{2\varepsilon\bar{\delta}^2(1-h)} \left[\int_{\bar{\delta}(1-\varepsilon)}^{2\varepsilon\bar{\delta}^2(1-h)} \bar{\delta}(1-h) d\delta - \int_{\bar{\delta}(1-\varepsilon)}^{\bar{\delta}(1-h)} \delta d\delta \right] \\
= \quad & \frac{\phi_3 l_2}{2\varepsilon\bar{\delta}^2(1-h)} \left[\bar{\delta}^2(\varepsilon - h)(1-h) - \frac{1}{2}\delta^2|_{\bar{\delta}(1-\varepsilon)}^{\bar{\delta}(1-h)} \right] \\
= \quad & \frac{\phi_3 l_2}{\bar{\delta}(1-h)} \left[\bar{\delta}^2(\varepsilon - h)(1-h) - \frac{1}{2}\bar{\delta}^2\{(1-h)^2 - (1-\varepsilon)^2\} \right] \\
= \quad & \frac{\phi_3 l_2}{2\varepsilon\bar{\delta}^2(1-h)} \left[\bar{\delta}^2(\varepsilon - h)(1-h) - \frac{1}{2}\bar{\delta}^2(\varepsilon - h)(2 - h - \varepsilon) \right] \\
= \quad & \frac{\phi_3 l_2}{4\varepsilon(1-h)} (\varepsilon - h)^2
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
1+i &= \frac{1}{2}\left[\begin{array}{c}\{1+\sigma^m[u'(\phi_3(2M))-1]\} \\ +\frac{\bar{\delta}}{\phi_3\psi_1}\{1+\sigma^a[u'(2A\phi_3 E(\psi_3)(1-h))-1+S(h)](1-h)\},\end{array}\right] \\
1+i &= \frac{1}{2}\left[\{1+\sigma^m[\frac{1}{\phi_3 2M}-1]\}+\frac{\bar{\delta}A}{\phi_3 M}\{1+\sigma^a[\frac{1}{2A\bar{\delta}(1-h)}-1+S(h)](1-h)\}\right] \\
1+i &= \frac{1}{2}\left[\{1+\sigma^m[\frac{1}{\phi_3 2M}-1]\}+\frac{\bar{\delta}A}{\phi_3 M}\{1+\sigma^a[\frac{1}{2A\bar{\delta}}-(1-h)+S(h)(1-h)]\}\right] \\
1+i &= \frac{1}{2}\left[\{1+\frac{\sigma^m}{\phi_3 2M}-\sigma^m\}+\frac{\bar{\delta}A}{\phi_3 M}\{1+\frac{\sigma^a}{2A\bar{\delta}}-\sigma^a(1-h)+\sigma^a S(h)(1-h)\}\right] \\
1+i &= \frac{1}{2}\left[\{1+\frac{\sigma^m}{\phi_3 2M}-\sigma^m\}+\frac{\bar{\delta}A}{\phi_3 M}\{1+\frac{\sigma^a}{2A\bar{\delta}}-\sigma^a(1-h)+\sigma^a S(h)(1-h)\}\right] \\
1+2i+\sigma^m &= \frac{1}{\phi_3 2M}\left[\sigma^m+2\bar{\delta}A\{1+\frac{\sigma^a}{2A\bar{\delta}}-\sigma^a(1-h)+\sigma^a S(h)(1-h)\}\right] \\
1+2i+\sigma^m &= \frac{1}{\phi_3 2M}\left[\sigma^m+2\bar{\delta}A+\sigma^a-2\bar{\delta}A\sigma^a(1-h)+2\bar{\delta}A\sigma^a S(h)(1-h)\right] \\
1+2i+\sigma^m &= \frac{1}{\phi_3 2M}\left[(1-\alpha)+2\bar{\delta}A\left(1-\sigma^a(1-h)[1-S(h)]\right)\right] \\
\Rightarrow \phi_3 &= \frac{1}{2M(1+2i+\sigma^m)}\left[(1-\alpha)+2\bar{\delta}A\left(1-\sigma^a(1-h)[1-S(h)]\right)\right] \\
\phi_3 &= \frac{1}{2M(1+2i+\sigma^m)}\left[(1-\alpha)+2\bar{\delta}A\left(1-\sigma^a[1-h-\frac{\bar{\delta}}{2}(\varepsilon-h)^2]\right)\right]
\end{aligned}
$$